Quantum Circuit Architecture

G. Chiribella, G. M. D’Ariano, and P. Perinotti

QUIT Group, Dipartimento di Fisica “A. Volta” and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via Bassi 6, I-27100 Pavia, Italy

(Received 25 December 2007; published 4 August 2008)

We present a method for optimizing quantum circuits architecture, based on the notion of a quantum comb, which describes a circuit board where one can insert variable subcircuits. Unexplored quantum processing tasks, such as cloning and storing or retrieving of gates, can be optimized, along with setups for tomography and discrimination or estimation of quantum circuits.

DOI: 10.1103/PhysRevLett.101.060401

PACS numbers: 03.65.Ta, 03.67.Lx

Quantum mechanics plays a crucial role in the technology of high precision and high sensitivity, e.g., in frequency standards [1], quantum lithography [2], two-photon microscopy [3], clock synchronization [4], and reference-frame transfer [5]. In these applications, the problem is to achieve high precision in (i) determining parameters and (ii) executing transformations that depend on unknown parameters. Since the parameters are generally encoded by a transformation, as in the whole class of quantum metrology problems [6], and since the estimation itself can be considered as a special case of transformation (with classical output), both tasks (i) and (ii) can be reduced to the general problem of executing a desired transformation depending on an unknown transformation. Taking into account the possibility of exploiting \( N \) uses of the unknown transformation, the problem is to build a quantum circuit that has \( N \) circuits as input, and achieves the desired transformation as an output. This is what we call a quantum circuit board.

A quantum circuit board is a network of gates in which there are \( N \) slots with open ports for the insertion of \( N \) variable subcircuits (see Fig. 1). Since generally it is impossible even in principle to achieve the desired transformation exactly, the main task here is to optimize the circuit board according to a given figure of merit. A typical example is the optimal cloning of an undisclosed transformation \( U \), which will be operated by a board with \( N \) slotted uses of \( U \), and achieving overall in-out transformation which is the closest possible to \( U^{\otimes M} \) with \( M > N \). We emphasize that generally the overall in-out transformation of the board and of the slotted circuits can be of any kind, including measurements and state-preparations, and the slotted transformations can be different from each other.

In previous literature, the only case of circuit-board optimization that has been considered is that of phase estimation [7]. In other applications, such as discrimination and estimation of unitary transformations with \( N \) uses, optimization has been carried out only for fixed architectures—i.e., with uses either in parallel [8,9], or in sequence [10]—since no systematic optimization method for variable architecture was available. On the other hand, the problem of deriving the optimal circuit board for channel tomography is still beyond the current possibilities of available optimization approaches.

In this Letter, we present a complete method for optimizing the architecture of quantum circuit boards. After providing a convenient description of circuit connectivity, we introduce the notion of quantum comb, which describes all possible transformations operated by a quantum circuit board, and generalizes the notion of quantum channel to the case where the inputs are quantum circuits, rather than quantum states. We then present the optimization method, based on the convex structure of the set of quantum combs. The method allows one to reduce the apparently untreatable problem of optimal circuit architecture to the optimization of a single positive operator with linear constraints. Since the positive operator summarizes all the relevant features of the circuit, our method automatically determines the optimal causal disposition of the variable slots. We will give several applications in which the present approach dramatically simplifies the solution of the problem.

A quantum circuit operates a transformation from input to output, and is graphically represented by a box with input and output wires symbolizing the respective quantum systems. Systems corresponding to different wires are generally different, and may also vary from input to output. Let us associate Hilbert spaces \( H_{\text{in}} \) (\( H_{\text{out}} \)) to all input (output) wires, and denote by \( \rho_{\text{in}} \) \( \rho_{\text{out}} \) the corresponding states. The action of the circuit is generally probabilistic; i.e., different in-out transformations can randomly occur, as in a measurement process. Each transformation is described by a linear map \( \rho_{\text{in}} \rightarrow \mathcal{C}(\rho_{\text{in}}) = k\rho_{\text{out}} \), with the proportionality factor \( 0 \leq k = \text{Tr}[\mathcal{C}(\rho_{\text{in}})] \leq 1 \) giving the probability that \( \mathcal{C} \) occurs on state \( \rho_{\text{in}} \). To describe a legitimate quantum transformation, the map \( \mathcal{C} : \text{Lin}(H_{\text{in}}) \rightarrow \text{Lin}(H_{\text{out}}) \) [11] has to be completely positive (CP) and trace

![FIG. 1 (color online). A quantum circuit board.](image)
nonincreasing. Trace-preserving maps—i.e., deterministic transformations—are called quantum channels. Notice that a map \( \mathcal{C} \), rather than representing a specific circuit, is univocally associated to the equivalence class of all circuits performing the same in-out transformation.

The linear map \( \mathcal{C} \) can be conveniently rewritten using the so-called “Choi-Jamiołkowski” representation [12], corresponding to the following one-to-one correspondence between linear maps \( \mathcal{C} : \Lin(H_{in}) \to \Lin(H_{out}) \) and linear operators \( C \in \Lin(H_{out} \otimes H_{in}) \) given by

\[
C = \text{Choi}(\mathcal{C}) := \mathcal{C} \otimes \mathcal{I}(\Omega)\Omega),
\]

(1)

where \( \mathcal{I} \) is the identity map, \(|\Omega\rangle = \sum_{n} |n\rangle n \rangle \in H_{in}^{\otimes 2} \), and \( T \) denotes transposition with respect to the orthonormal basis \(|n\rangle \rangle \) for \( H_{in} \). The map \( \mathcal{C} \) is CP if and only if the operator \( C \)—called Choi operator—is positive [13].

Two quantum circuits can be connected in all the ways allowed by the physical matchings between input and output wires (see, e.g., Fig. 2, where the wires labeled \( d \) are connected): a connection will result in the composition of the corresponding CP maps, and hence of the corresponding Choi operators. Since building a quantum network means connecting many circuits, it is crucial to have a handy way to describe circuit connectivity with minimum overhead of notation. We provide here three simple rules that accomplish this goal:

**Rule 1 (Labelling)** Each quantum wire is marked with a different label, except for wires that are connected, which are identified with the same label.

**Rule 2 (Multiplication)** The multiplication of two Choi operators \( A \in \Lin(H_{a,b,c,d}) \) and \( B \in \Lin(H_{d,e,f,g}) \) is regarded in the tensor fashion, i.e., \( AB = (A \otimes I_{e,f,g})(I_{a,b,c} \otimes B) \).

**Rule 3 (Composition)** The composition of two circuits with Choi operators \( A \) and \( B \)—acting on Hilbert spaces labeled according to Rule 1—yields a new circuit with Choi operator \( C \) given by the link product

\[
C = A \ast B = \text{Tr}_{J}[A^{\theta_j} B],
\]

(3)

\( \theta_j \) denoting partial transposition over the Hilbert space \( J \) of the connected wires, and the multiplication in square brackets following Rule 2.

Rule 3 follows from Eqs. (1) and (2). Notice that due to invariance of trace under cyclic permutations, the link product is commutative: \( A \ast B = B \ast A \). Using it, the action of a linear map \( \mathcal{C} \) on a state \( \rho \) in Eq. (2) can be rewritten as \( \mathcal{C}(\rho) = C \ast \rho \). Assembling many circuits \( C_1, C_2, \ldots, C_k \) yields a quantum network whose Choi operator is simply given by \( C = C_1 \ast C_2 \ast \ldots \ast C_k \).

We are now ready to treat quantum circuit boards. To start with, we consider the case of a deterministic circuit board, i.e., a network of quantum channels with \( N \) open slots for the insertion of variable subcircuits. It is clear that by reshuffling and stretching the internal wires, any circuit board can be reshaped in the form of a “comb,” with an ordered sequence of slots, each between two successive teeth, as in Fig. 3. The order of the slots is the causal order induced by the flow of quantum information in the circuit board. We label the input systems (entering the board) with even numbers \( 2n \), and the corresponding output systems (exiting the board) with odd numbers \( 2n + 1 \), with \( n \) ranging from \( 0 \) to \( N \).

A quantum comb with \( N \) slots is clearly equivalent to a concatenation of \( N + 1 \) channels with memory, which is in turn equivalent to a causal network, namely, a network where the quantum state of the output systems up to time \( n \) does not depend on the state of the input systems at later times \( n' > n \), with \( n, n' \in \{0, 1, \ldots, N\} \) [14]. The causal network can be easily obtained by redrawing the comb as an equivalent circuit with all inputs on the left and all outputs on the right, as in Fig. 4. We define the Choi operator of a quantum comb as the Choi operator \( R \) of the corresponding causal network. In terms of the Choi operator \( R \), causality is equivalent to a set of linear constraints

\[
\text{Tr}_{2n+i}[R^{(n)}] = I_{2n} \otimes R^{(n-1)}, \quad n = 0, \ldots, N, \quad R^{(N)} = R, \quad R^{(-1)} = 1,
\]

(4)

where \( \text{Tr}_{2n+1} \) denotes the partial trace over the Hilbert space of the
space $H_{2n+1}$ of the output wire labeled $2n+1$, $I_{2n}$ the identity operator over the Hilbert space $H_{2n}$ of the input wire labeled $2n$, $R^{(n)} = \text{Choi}(C^{(n)})$, and $C^{(n)}$ is the map of the $(n+1)$-subnetwork from the first $n+1$ inputs to the first $n+1$ outputs. Precisely, we have the following:

**Theorem 1.**—Every positive operator $0 \leq R \in \text{Lin}(\otimes_{j=0}^{2N+1} H_j)$ satisfying the linear constraints (4), is the Choi operator of a deterministic quantum comb.

**Proof.**—By definition, it is enough to show that any operator $R \geq 0$ normalized as in Eq. (4) is the Choi operator of a causal network. A causal network with $N+1$ input/output pairs is described by a family of channels $H_C$ corresponding to the causal network, with all inputs on the left and all outputs on the right. The Choi operator of a comb is the Choi operator of the corresponding causal network.

The first is the optimal universal cloning of unitary transformations, i.e., the problem of designing a quantum board that optimally achieves the $N \rightarrow M$ cloning of an unknown unitary $U \in SU(d)$ in dimension $d$. The board has $N$ slots containing $N$ identical uses of the unknown unitary $U$ and performs a transformation which is the closest possible to $U^\otimes M$. Evaluation for $1 \rightarrow 2$ cloning [16] using as figure of merit the channel fidelity averaged over all possible unitaries leads to optimal value $F = (d + \sqrt{d^2 - 1})/d^3$, significantly higher than the classical threshold reached by the optimal estimation of a unitary $F_{\text{est}} = 6/d^4$ for $d > 2$, $F_{\text{est}} = 5/16$ for qubits, thus showing the advantage of coherent quantum information processing over any classical cloning strategy.

A second interesting application is the storage and retrieval of an undisclosed unitary transformation $U$ from $N$ uses, also called optimal quantum-algorithm learning. The problem arises from the need of running an undisclosed algorithm (available for $N$ uses) on an input state $\psi$ which will be available at later time. To this purpose, one can slot the $N$ uses of $U$ in a quantum circuit board, put the output state of the board in a quantum memory, and, when the input state will be available, use the memory to recover the unitary. The series storing or retrieving is represented by the quantum comb in Fig. 6, which can be cut into two

**FIG. 4** (color online). Each quantum comb is equivalent to a causal network, with all inputs on the left and all outputs on the right. The Choi operator of a comb is the Choi operator of the corresponding causal network.

**FIG. 5** (color online). A quantum comb realizes different transformations of quantum circuits, namely, it can send (a) a series of channels into a channel, (b) a series of channels into a comb, or (c) an input comb into an output comb.

**FIG. 6** (color online). Quantum-algorithm learning. One wants to run an undisclosed unitary $U$ on a quantum state $\psi$ which is available after the lapse of time in which the uses of $U$ are available.
In conclusion, we introduced a new method for optimization of quantum networks and illustrated its effectiveness in several applications in which our theory solves problems that could not even be addressed otherwise. As main examples, comb theory allows us to optimize cloning, storage retrieving, discrimination, estimation, and tomographic characterization of quantum circuits.

[11] Here we consider only finite dimensions and denote the linear space of operators on $H$ as $\text{Lin}(H)$.
[15] This can be proven within an axiomatic introduction of combs and supermaps, where a realization theorem holds stating that any CP supermap from combs to combs has a physical scheme corresponding to another comb.
[17] Introducing a classical register with orthogonal states $|i\rangle \in \mathcal{H}_c$, we can define $\tilde{R} = \sum_i R_i \otimes |i\rangle\langle i|$, which is the Choi operator of a deterministic comb with $\mathcal{H}_{N+1} := \mathcal{H}_N \otimes \mathcal{H}_c$. The comb corresponding to $R_i$ is then obtained after applying the comb of $\tilde{R}$, by measuring the register on the basis $\{|i\rangle\}$ and post-selecting outcome $i$.