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No-signalling, dynamical independence and the local observability principle

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Abstract
Within a general operational framework I show that a-causality at a distance of 'local actions' (the so-called no-signalling) is a direct consequence of commutativity of local transformations, i.e. of dynamical independence. On the other hand, the tensor product of quantum mechanics is not just a consequence of such dynamical independence, but needs in addition the local observability principle.

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1. Introduction

The quantum correlations due to entanglement are instantaneous. At first sight this may seem useful for superluminal communications, e.g. in a communication scheme in which Alice, using a singlet state entangled with Bob's spin, communicates to him a bit value $b = 0, 1$ by measuring either σ_z or σ_x, respectively, and then Bob tries to determine if its local spin state is an eigenstate of σ_z or of σ_x. Such a communication scheme was indeed considered in [1]), where a strategy for discriminating Bob's non-orthogonal states has been devised based on cloning states into multiple copies via stimulated emission of radiation. However, the possibility of cloning quantum states was ruled out in [2–5] (for a history of the no-cloning theorem see [6]), where it was shown that perfect cloning is impossible as a consequence of linearity of quantum-mechanical transformations, and, as a consequence of the no-cloning theorem, it is impossible to discriminate with certainty among non-orthogonal states [7].

From the point of view of proving *no-signalling*, i.e. the impossibility of superluminal communications, the no-cloning argument, however, is incomplete, since it does not rule out the possibility of information transmission by other means, e.g. by approximate cloning [8–11], or probabilistic cloning [12]. For example, we know that in some cases we can discriminate perfectly among nonorthogonal states [13], however, with some probability of inconclusive outcome: could not this be used to achieve a superluminal communication with

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some probability? Who guarantees that any quantum operation performed by Alice and Bob cannot be used to make a superluminal communication using some entangled state?

Since neither no-cloning, nor no-state-discrimination impossibility theorems logically imply no-signalling, an independent rigorous proof of no-signalling is in order, and, indeed, several authors [14–19] have analysed the issue and proved no-signalling. The first proof that non-locality of quantum mechanics cannot lead to superluminal transmission of information has been given in [14, 15], and was then generalized to any trace-preserving quantum operation in [19]. A proof of the local state invariance for trace-preserving quantum operations has also been given in [20].

The 'peaceful coexistence' [21] between quantum non-locality and special relativity has intrigued many physicists for years, on whether the no-signalling condition plays a more fundamental role, e.g. it could be used as an axiom for deriving quantum mechanics itself. In this line of thought some basic features of quantum mechanics have been analysed, such as no-cloning itself. For example, the no-signalling constraint has been used to derive upper bounds for the fidelity of cloning transformations [22–25]. Later, however, the existence of a connection between approximate cloning and the no-signalling has been ruled out [19], and it has been shown that the no-signalling constraint on its own is not sufficient to prevent a transformation from surpassing the known optimal cloning bounds. More specifically, in [26] the possibility of using no-signalling as an axiom of quantum mechanics has been considered again, arguing that, once the Born rule is assumed, the linearity of quantum mechanics can be derived from the no-signalling condition. A big step forward in understanding the axiomatic role played by no-signalling in quantum mechanics has been done in [27]. There it has been shown that, at the purely statistical level, there exist in principle super-quantum correlations that violate the quantum bound (such as the Tsirelson's bound [28] for the CHSH correlation [29]) without anyway violating the no-signalling condition. Therefore, it is possible in principle to have a non-locality that is even stronger than the quantum one, however, still without violating the no-signalling.

The above considerations and the past research history on no-signalling suggest seeking more precise logical connections between seemingly related issues such as locality, causality, dynamical independence and statistical independence, within a general purely operational framework. In this paper, I will show that starting from a very general and comprehensive definition of *action* by an agent in a communication scenario, the no-signalling is a direct consequence of commutativity of local transformations, i.e. of dynamical independence. In the process, I will also give an alternative very general proof of no-signalling in quantum mechanics. On the other hand, I will show that the tensor product of quantum mechanics (which leads to no-signalling) is not just a consequence of dynamical independence, but needs an additional hypothesis, which is the *local observability principle* [30]. This plays a crucial operational role in reducing the experimental complexity for experiments on composite systems, reconciling holism with reductionism in a non-local theory. For a complete account on the operational framework used in the present and a related axiomatic derivation of quantum mechanics, the reader is addressed to [30].

2. Operational derivation of no-signalling from dynamical independence

In a purely operational framework, beyond physical theories, in analysing a communication scenario we need precise definitions for *action* of a transmitting agent, *locality* of actions and *dynamical independence*. As we will see, the dynamical independence is essentially synonym of existence of local actions, and locality of actions is synonym of commutativity of transformations, which in turn leads to no-signalling.

2.1. Action and state

Definition 1 (action)**.** *The action on a object system (due to an agent producing an interaction of the object with an apparatus) leads to an object transformation drawn from a set of possible transformations, each one occurring with some probability.*

According to our definition, the action is identified with a set $\mathbb{A} \equiv \{\mathscr{A}_j\}$ of possible transformations $\mathscr{A}_j$ that can occur on the object system. In an ideal situation the apparatus signals which transformation actually occurred, and the agent has perfect knowledge of all details of each transformation. The agent cannot control which transformation occurs, but he can decide which action to perform, namely he can choose the set of possible transformations $\mathbb{A} = \{\mathscr{A}_j\}$. For example, in an Alice&Bob communication scenario Alice encodes the different bit values by choosing between two actions $\mathbb{A} = \{\mathscr{A}_j\}$ and $\mathbb{B} = \{\mathscr{B}_j\}$ corresponding to two different sets of transformations $\{\mathscr{A}_j\}$ and $= \{\mathscr{B}_j\}$. The agent has control on the transformation itself only in the special case when the transformation $\mathscr{A}$ is deterministic. In the following, wherever we consider a nondeterministic transformation $\mathscr{A}$, we always regard it in the context of an action $\mathbb{A} = \{\mathscr{A}, \mathscr{B}\}$, namely assuming that there exists a complementary transformation $\mathscr{B}$ such that either $\mathscr{A}$ or $\mathscr{B}$ occurs.

Definition 2 (state)**.** *A* state *is a probability rule for transformations.*

Therefore, ω is a state means that $\omega(\mathscr{A})$ is a map from the set of all possible transformations to [0, 1] satisfying the completeness condition $\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$. We will take the identical transformation $\mathscr{I}$ as *no-action* with $\omega(\mathscr{I}) = 1$. In the following for a given physical system we will denote by $\mathfrak{S}$ the set of all possible states and by $\mathfrak{T}$ the set of all possible transformations.

2.2. Dynamics as conditioning

State conditioning. When composing two transformations $\mathscr{A}$ and $\mathscr{B}$, the probability $p(\mathscr{B}|\mathscr{A})$ that $\mathscr{B}$ occurs conditional on the previous occurrence of $\mathscr{A}$ is given by the Bayes rule for conditional probabilities $p(\mathscr{B}|\mathscr{A}) = \omega(\mathscr{B} \circ \mathscr{A})/\omega(\mathscr{A})$. This sets a new probability rule corresponding to the notion of *conditional state* $\omega_{\mathscr{A}}$ which gives the probability that a transformation $\mathscr{B}$ occurs knowing that the transformation $\mathscr{A}$ has occurred on the physical system in the state ω, namely $\omega_{\mathscr{A}} \doteq \omega(\cdot \circ \mathscr{A})/\omega(\mathscr{A})$ (in the following we will make extensive use of the functional notation with the central dot corresponding to a variable transformation). One can see that the present definition of 'state' leads to the identification *state-evolution≡state-conditioning*, entailing a *linear action of transformations on states* (apart from normalization) $\mathscr{A}\omega := \omega(\cdot \circ \mathscr{A})$: this is the same concept of *operation* that we have in quantum mechanics. Therefore, in the present context linearity of evolution is just a consequence of the fact that the evolution of states is pure state-conditioning: this will include also the deterministic case $\mathscr{U}\omega = \omega(\cdot \circ \mathscr{U})$ of transformations $\mathscr{U}$ with $\omega(\mathscr{U}) = 1$ for all states ω—the analogous of quantum unitary evolutions and channels.

Dynamical and informational equivalence. From the Bayes conditioning it follows that we can define two complementary types of equivalences for transformations: the *dynamical* and *informational* equivalences. The transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are *dynamically equivalent* when $\omega_{\mathscr{A}_1} = \omega_{\mathscr{A}_2} \forall\omega \in \mathfrak{S}$, whereas they are *informationally equivalent* when $\omega(\mathscr{A}_1) = \omega(\mathscr{A}_2) \forall\omega \in \mathfrak{S}$. The two transformations are then completely equivalent when they are both dynamically and informationally equivalent, corresponding to the identity $\omega(\mathscr{B} \circ \mathscr{A}_1) = \omega(\mathscr{B} \circ \mathscr{A}_2), \forall\omega \in \mathfrak{S}, \forall\mathscr{B} \in \mathfrak{T}$. We call effect an informational equivalence class of transformations (this is the same notion introduced by Ludwig [31]). In the following we will denote effects with the underlined symbols $\underline{\mathscr{A}}$, $\underline{\mathscr{B}}$, and we will write $\mathscr{A}_0 \in \underline{\mathscr{A}}$ meaning that 'the transformation $\mathscr{A}_0$ belongs to the equivalence class $\underline{\mathscr{A}}$', or '$\mathscr{A}_0$ corresponds to the effect $\underline{\mathscr{A}}$', or '$\mathscr{A}_0$ is informationally equivalent to $\mathscr{A}$'. Since, by definition one has $\omega(\mathscr{A}) \equiv \omega(\underline{\mathscr{A}})$, we will legitimately write $\omega(\underline{\mathscr{A}})$ instead of $\omega(\mathscr{A})$. Similarly, one has $\omega_{\mathscr{A}}(\mathscr{B}) \equiv \omega_{\mathscr{A}}(\underline{\mathscr{B}})$, which implies that $\omega(\mathscr{B} \circ \mathscr{A}) = \omega(\underline{\mathscr{B}} \circ \mathscr{A})$, which gives the chaining rule $\underline{\mathscr{B}} \circ \mathscr{A} \in \underline{\mathscr{B} \circ \mathscr{A}}$ corresponding to the 'Heisenberg picture' evolution of transformations acting on effects (note that in this way transformations act from the right on effects). Now, by definitions effects are linear functionals over states with range [0, 1], and, by duality, we have a convex structure over effects. We will denote the convex set of effects by $\mathfrak{P}$.

2.3. The structure of transformations

Addition of transformations. The fact that we necessarily work in the presence of partial knowledge about both object and apparatus corresponds to the possibility of incomplete specification of both states and transformations, entailing the convex structure on states and the addition rule for *coexistent transformations*, namely for transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ for which $\omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \leqslant 1, \forall\omega \in \mathfrak{S}$ (i.e. transformations that can in principle occur in the same action). The addition of the two coexistent transformations is the transformation $\mathscr{S} = \mathscr{A}_1 + \mathscr{A}_2$ corresponding to the event $e = \{1, 2\}$ in which the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred, but does not specify which one. Such transformation is specified by the informational and dynamical equivalence classes $\forall\omega \in \mathfrak{S}$: $\omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2)$ and $(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1\omega + \mathscr{A}_2\omega$. Clearly the composition '$\circ$' of transformations is distributive with respect to the addition '+'. We will also denote as $\mathscr{S}(\mathbb{A}) := \sum_{\mathscr{A}_j \in \mathbb{A}} \mathscr{A}_j$ the deterministic transformation $\mathscr{S}(\mathbb{A})$ corresponding to the sum of all possible transformations $\mathscr{A}_j$ in $\mathbb{A}$. We can also define the multiplication $\lambda\mathscr{A}$ of a transformation $\mathscr{A}$ by a scalar $0 \leqslant \lambda \leqslant 1$ as the transformation which is dynamically equivalent to $\mathscr{A}$, but occurs with rescaled probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$. Now, since for every couple of transformation $\mathscr{A}$ and $\mathscr{B}$ the transformations $\lambda\mathscr{A}$ and $(1-\lambda)\mathscr{B}$ are coexistent for $0 \leqslant \lambda \leqslant 1$, the set of transformations also becomes a convex set. Moreover, since the composition $\mathscr{A} \circ \mathscr{B}$ of two transformations $\mathscr{A}$ and $\mathscr{B}$ is itself a transformation and there exists the identical transformation $\mathscr{I}$ satisfying $\mathscr{I} \circ \mathscr{A} = \mathscr{A} \circ \mathscr{I} = \mathscr{A}$ for every transformation $\mathscr{A}$, the transformations make a semigroup with identity, i.e. a *monoid*. Therefore, the set of physical transformations is a convex monoid.

It is obvious that we can extend the notions of coexistence, sum and multiplication by a scalar from transformations to effects via equivalence classes.

2.4. Dynamical independence and local state

A purely dynamical notion of *independent systems* coincides with the possibility of performing local actions. More precisely, we define

Definition 3 (dynamical independence)**.** *Two physical systems are independent if on the two systems 1 and 2 we can perform local actions $\mathbb{A}^{(1)}$ and $\mathbb{A}^{(2)}$ whose transformations commute each other (i.e. $\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}, \forall\mathscr{A}^{(1)} \in \mathbb{A}^{(1)}, \forall\mathscr{B}^{(2)} \in \mathbb{B}^{(2)}$).*

Note that the above definition of independent systems is purely dynamical, in the sense that it does not contain any statistical requirement, such as the existence of factorized states. Indeed, the present notion of dynamical independence is so minimal that it can be satisfied not only by the quantum tensor product, but also by the quantum direct sum. As we will see in the following, it is the local observability principle of Postulate 1 which will select the tensor product. In the following, when dealing with more than one independent system, we will denote local transformations as ordered strings of transformations as follows $\mathscr{A},\mathscr{B},\mathscr{C},\ldots := \mathscr{A}^{(1)}\circ\mathscr{B}^{(2)}\circ\mathscr{C}^{(3)}\circ\ldots$. The notion of independent systems now entails the notion of *local state*—the equivalent of partial trace in quantum mechanics.

Definition 4 (local state)**.** *For two independent systems in a joint state Ω, we define the local state $\Omega|_1$ of system 1 as the probability rule $\Omega|_1(\mathscr{A}) \doteq \Omega(\mathscr{A},\mathscr{I})$ of the joint state Ω with a local transformation $\mathscr{A}$ only on the system 1 and with system 2 untouched.*

Clearly, the above notion can be symmetrically defined for system 2, and can be trivially extended to any number of independent systems, with the local state $\Omega|_n$ of the nth system representing the probability rule of the joint state in which all systems are left untouched apart from system n.

3. The no-signalling theorem

We are now in a position to prove the general no-signalling theorem.

Theorem 1 (no-signalling)**.** *Any local action on a system does not affect another independent system. More precisely, any local action on a system is equivalent to the identity transformation when viewed from another independent system. In equations one has*

$$\forall\Omega\in\mathfrak{S}^{\times 2},\ \forall\mathbb{A},\qquad \Omega_{\mathscr{S}(\mathbb{A}),\mathscr{I}}|_2 = \Omega|_2. \tag{1}$$

Proof. Since the two systems are dynamically independent, for every two local transformations one has $\mathscr{A}^{(1)}\circ\mathscr{A}^{(2)} = \mathscr{A}^{(2)}\circ\mathscr{A}^{(1)}$, which implies that $\Omega(\mathscr{A}^{(1)}\circ\mathscr{A}^{(2)}) = \Omega(\underline{\mathscr{A}}^{(1)}\circ\mathscr{A}^{(2)}) = \Omega(\mathscr{A}^{(1)}\circ\underline{\mathscr{A}}^{(2)}) \equiv \Omega(\underline{\mathscr{A}}^{(1)},\underline{\mathscr{A}}^{(2)})$. By definition, for $\mathscr{B}\in\mathfrak{T}$ one has $\Omega|_2(\mathscr{B}) = \Omega(\mathscr{I},\mathscr{B})$, and using the addition rule for transformations and recalling the definition of $\mathscr{S}(\mathbb{A})$, one has

$$\Omega(\mathscr{S}(\mathbb{A}),\mathscr{B}) = \Omega([\mathscr{S}(\mathbb{A})]_{\mathrm{inf}},\underline{\mathscr{B}}) = \Omega(\underline{\mathscr{I}},\underline{\mathscr{B}}) =: \Omega|_2(\mathscr{B}), \tag{2}$$

where $[\cdot]_{\mathrm{inf}}$ denotes the informational equivalence class. On the other hand, we have

$$\Omega_{\mathscr{S}(\mathbb{A}),\mathscr{I}}|_2(\mathscr{B}) = \Omega((\mathscr{I},\mathscr{B})\circ(\mathscr{S}(\mathbb{A}),\mathscr{I})) = \Omega(\mathscr{S}(\mathbb{A}),\mathscr{B}), \tag{3}$$

namely the statement. □

Note how the no-signalling is a mere consequence of our minimal notion of dynamical independence in definition 3. Note also the consistency with the dynamical part of the definition of addition of coexistent transformations, i.e. conditioning

$$\begin{aligned}\Omega_{\mathscr{S}(\mathbb{A}),\mathscr{I}}|_2(\mathscr{B}) &= \Omega_{\mathscr{S}(\mathbb{A}),\mathscr{I}}(\mathscr{I},\mathscr{B}) = \sum_{\mathscr{A}_j\in\mathbb{A}}\Omega_{\mathscr{A}_j,\mathscr{I}}(\mathscr{I},\mathscr{B})\frac{\Omega(\mathscr{A}_j,\mathscr{I})}{\sum_{\mathscr{A}_j\in\mathbb{A}}\Omega(\mathscr{A}_j,\mathscr{I})}\\ &= \sum_{\mathscr{A}_j\in\mathbb{A}}\frac{\Omega(\mathscr{A}_j,\mathscr{B})}{\Omega(\mathscr{A}_j,\mathscr{I})}\frac{\Omega(\mathscr{A}_j,\mathscr{I})}{\Omega(\mathscr{I},\mathscr{I})} = \sum_{\mathscr{A}_j\in\mathbb{A}}\Omega(\mathscr{A}_j,\mathscr{B}) = \Omega(\mathscr{I},\mathscr{B}).\end{aligned} \tag{4}$$

Corollary 1. *One has the logical equivalence*

$$\Omega(\mathscr{A},\mathscr{I}) = 1 \Longleftrightarrow \Omega(\mathscr{A},\mathscr{B}) = \Omega(\mathscr{I},\mathscr{B}),\qquad \forall\mathscr{B}\in\mathfrak{T}. \tag{5}$$

Proof. The implication from the left to the right is trivial. To prove the reverse implication, just consider another transformation $\mathscr{A}^{\#}$ to complete an action $\mathbb{A} = \{\mathscr{A},\mathscr{A}^{\#}\}$. Now $0 = \Omega(\underline{\mathscr{A}}^{\#},\underline{\mathscr{I}}) = \Omega(\underline{\mathscr{A}}^{\#},\underline{\mathscr{B}}) + \Omega(\underline{\mathscr{A}}^{\#},\underline{\mathscr{B}}^{\#})$ which implies that $\Omega(\underline{\mathscr{A}}^{\#},\underline{\mathscr{B}}^{\#}) = \Omega(\underline{\mathscr{A}}^{\#},\underline{\mathscr{B}}) = 0$. This implies that $\Omega(\underline{\mathscr{I}},\underline{\mathscr{B}}) = \Omega(\underline{\mathscr{A}},\underline{\mathscr{B}}) + \Omega(\underline{\mathscr{A}}^{\#},\underline{\mathscr{B}}) = \Omega(\underline{\mathscr{A}},\underline{\mathscr{B}})$. □

Assessing the truth of statement (5) implies no-signalling, since if $\Omega(\mathscr{S}(\mathbb{A}),\mathscr{I}) = 1 \Longrightarrow \Omega(\mathscr{S}(\mathbb{A}),\mathscr{B}) = \Omega(\mathscr{I},\mathscr{B})$, i.e. $\Omega(\mathscr{S}(\mathbb{A}),\mathscr{B}) = \Omega_2(\mathscr{B})\forall\mathscr{B}\in\mathfrak{T}$.

4. The quantum version of no-signalling theorem

Since assessing the truth of statement (5) implies the no-signalling, in order to prove no-signalling in quantum mechanics we just need to prove validity of (5) in the quantum case. For this purpose, we need a simple technical lemma that is reported in appendix A. We can then prove the quantum version of no-signalling.

Theorem 2 (quantum version of corollary 1)**.** *For any positive operator $R \in \mathsf{H}_A \otimes \mathsf{H}_B$ and any generally trace-decreasing quantum operation $\mathscr{M}$ which acts locally on states on H_A, one has*

$$\mathrm{Tr}[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}[R] \quad \Longleftrightarrow \quad \mathrm{Tr}_1[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}_1[R]. \tag{6}$$

Proof. That the identity $\mathrm{Tr}_1[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}_1[R]$ implies $\mathrm{Tr}[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}[R]$ is obvious. The converse implication is not obvious. Therefore, assume that

$$\mathrm{Tr}[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}[R]. \tag{7}$$

Invariance of trace under cyclic permutation gives

$$\mathrm{Tr}_1[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}_1[(K \otimes I)R], \tag{8}$$

where $K = \mathscr{M}^{\intercal}(I)$, and $\mathscr{M}^{\intercal}$ denotes the quantum operation on the Heisenberg picture. Hence, one has

$$\mathrm{Tr}_1[\mathscr{M} \otimes \mathscr{I}(R)] = \mathrm{Tr}_1[R] + \mathrm{Tr}_1\{[(K - I) \otimes I]R\} \equiv \mathrm{Tr}_1[R]. \tag{9}$$

In fact, due to equation (7), one has

$$\mathrm{Tr}\{[(I - K) \otimes I]R\} = 0, \tag{10}$$

but according to lemma 1 in appendix A, the operator $\mathrm{Tr}_1\{[(I-K)\otimes I]R\}$ is positive, whence, being trace-less, it must be identically zero. □

5. The tensor product and the local observability principle

The tensor product realization of dynamically independent systems in quantum mechanics does not follow just from the general definition of dynamical independence. Indeed, definition 3 does not exclude the quantum-mechanical realization in terms of direct sum, instead of tensor product (see appendix B). One way of excluding the direct sum realization is to consider the existence of states for which the probability factorizes e.g. $\Omega(\mathscr{A}, \mathscr{B}) = \omega_1(\mathscr{A})\omega_2(\mathscr{B})$; however, this would lead to a definition of independence that is not purely dynamical, but also statistical. A way to exclude the direct sum in a purely dynamical way is to introduce the following *local observability principle*

Postulate 1 (local observability principle)**.** *For every composite system there exist informationally complete observables made only of local informationally complete observables.*

We recall the definition of informationally complete observable.

Definition 5 (informationally complete observable)**.** *An observable $\mathbb{L} = \{l_i\}$ is informationally complete if each effect can be written as a linear combination of elements of $\mathbb{L}$, namely for each effect l there exist coefficients $c_i(l)$ such that*

$$l = \sum_i c_i(l) l_i. \tag{11}$$

We call the informationally complete observable *minimal* when its effects are linearly independent.

As a consequence of the duality between the convex set of states and that of effect, one has the identity of their affine dimensions $\dim(\mathfrak{S}) = \dim(\mathfrak{P}) - 1$ (the missing dimension is due to the normalization condition for states).

The local observability principle plays a crucial operational role, since it reduces enormously the experimental complexity, by guaranteeing that only local (although jointly executed!) experiments are sufficient to retrieve a complete information of a composite system, including all correlations between the components. The principle reconciles holism with reductionism in a non-local theory, in the sense that we can observe a holistic nature in a reductionistic way—i.e. locally. The principle implies the following identity

Theorem 3. *The affine dimension of the convex set of states $\mathfrak{S}_{12}$ of a composed system can be written in terms of the affine dimensions of the convex sets of states $\mathfrak{S}_1$ and $\mathfrak{S}_2$ of the component systems as*

$$\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2). \tag{12}$$

Proof. We can first prove that the right-hand side of equation (12) is an upper bound for the left-hand side. Indeed, as we have seen, by duality between $\mathfrak{S}$ and $\mathfrak{P}$ the number of outcomes of a minimal informationally complete observable is given by $\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$. Now, consider a global informationally complete measurement made of two local minimal informationally complete observables measured jointly. It has number of outcomes $[\dim(\mathfrak{S}_1) + 1][\dim(\mathfrak{S}_2) + 1]$. However, we are not guaranteed that the joint observable is itself minimal, whence the right-hand side of equation (12) is just an upper bound.

The opposite bounding can be easily proved by considering that a global informationally incomplete measurement made of minimal local informationally complete measurements should belong to the linear span of a minimal global informationally complete measurement. □

Identity (12) is the same that we have in quantum mechanics as a consequence of the tensor product structure. In fact one has $\dim(\mathfrak{S}) = \dim(\mathsf{H})^2 - 1$, and $\dim(\mathsf{H}_{12}) = \dim(\mathsf{H}_1)\dim(\mathsf{H}_2)$, which gives $\dim(\mathfrak{S}_{12}) + 1 = [\dim(\mathfrak{S}_1) + 1][\dim(\mathfrak{S}_2) + 1]$. Therefore, the tensor product is not a consequence of dynamical independence in definition 3, but follows from the local observability principle.

6. Conclusion

We have considered a very general operational framework, and proved that the so-called *no-signalling* (a-causality at a distance of 'local actions') is a direct consequence of dynamical independence of systems. We have seen that the concept of purely dynamical independence can only be defined in terms of commutativity of local transformations. Hence, as such, dynamical independence is compatible with both tensor product and direct sum of operator algebras in quantum mechanics. On the other hand, the tensor product description of independent systems in quantum mechanics operationally follows from the additional requirement of local observability. The local observability principle plays a crucial operational role, reconciling holism with reductionism in a non-local theory, allowing us to observe a holistic nature in a reductionistic way—i.e. locally.

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Appendix A. Technical lemma

Lemma 1. *For $A \geqslant 0$ operator on H_A and $R \geqslant 0$ operator on $\mathsf{H}_a \otimes \mathsf{H}_B$ one has*

$$\mathrm{Tr}_1[(A \otimes I)R] \geqslant 0. \tag{A.1}$$

Proof. For any vector $|\varphi\rangle \in \mathsf{H}_A$ one has $\mathrm{Tr}_1[(|\varphi\rangle\langle\varphi| \otimes I)R] \geqslant 0$, since for any vector $|\phi\rangle \in \mathsf{H}_B$ one has

$$\langle\phi|\mathrm{Tr}_1[(|\varphi\rangle\langle\varphi| \otimes I)R]|\phi\rangle = (\langle\phi| \otimes \langle\varphi|)R(|\phi\rangle \otimes |\varphi\rangle) \geqslant 0, \tag{A.2}$$

due to positivity of R. Then, the statement follows by considering a spectral decomposition of A, namely

$$\mathrm{Tr}_1[(A \otimes I)R] = \sum_n a_n \mathrm{Tr}_1[(|\varphi_n\rangle\langle\varphi_n| \otimes I)R] \geqslant 0. \tag{A.3}$$

□

Appendix B. The direct-sum dynamical independence

For a direct sum pair of systems, a local transformation on system 1 works on a joint state as $\mathscr{A}^{(1)} = \mathscr{A}_+ \oplus p_{\mathscr{A}} \mathscr{I}_-$, namely, on a joint state Ω corresponding to $\rho_+ \oplus \rho_-$ one has

$$\Omega(\mathscr{A}, \mathscr{I}) = \mathrm{Tr}[\mathscr{A}_+(\rho_+)] + p_{\mathscr{A}} \mathrm{Tr}[\rho_-]. \tag{B.1}$$

Any couple of local transformations on the two 'systems' commute, since

$$\begin{aligned} \mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} &= (\mathscr{A}_+ \oplus p_{\mathscr{A}} \mathscr{I}_-)(p_{\mathscr{B}} \mathscr{I}_+ \oplus \mathscr{B}_-) = p_{\mathscr{B}} \mathscr{A}_+ \oplus p_{\mathscr{A}} \mathscr{B}_- \\ &= (p_{\mathscr{B}} \mathscr{I}_+ \oplus \mathscr{B}_-)(\mathscr{A}_+ \oplus p_{\mathscr{A}} \mathscr{I}_-) = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}. \end{aligned} \tag{B.2}$$

The probability rule of a joint state on local transformations is

$$\Omega(\mathscr{A}, \mathscr{B}) = \mathrm{Tr}[p_{\mathscr{B}} \mathscr{A}_+ \oplus p_{\mathscr{A}} \mathscr{B}_-(\rho)] = p_{\mathscr{B}} \mathrm{Tr}[\mathscr{A}_+(\rho_+)] + p_{\mathscr{A}} \mathrm{Tr}[\mathscr{B}_-(\rho_-)], \tag{B.3}$$

which gives the implication in the statement of Corollary 1—i.e. implying no-signalling—since $\Omega(\mathscr{A}, \mathscr{I}) = 1$ is satisfied only for $p_{\mathscr{A}} = 1$ and trace-preserving $\mathscr{A}_+$, which then implies $\Omega(\mathscr{A}, \mathscr{B}) = p_{\mathscr{B}} \mathrm{Tr}[\rho_+] + \mathrm{Tr}[\mathscr{B}_-(\rho_-)] \equiv \Omega(\mathscr{I}, \mathscr{B})$. Note how also state conditioning is consistently defined

$$\Omega_{\mathscr{A},\mathscr{I}}(\mathscr{B}) = \frac{p_{\mathscr{B}} \mathrm{Tr}[\mathscr{A}(\rho_+)] + p_{\mathscr{A}} \mathrm{Tr}[\mathscr{B}(\rho_-)]}{\mathrm{Tr}[\mathscr{A}(\rho_+)] + p_{\mathscr{A}} \mathrm{Tr}[\rho_-]}. \tag{B.4}$$

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