Operational Axioms for C*-algebra Representation of Transformations

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Abstract. It is shown how a C*-algebra representation of the transformations of a physical system can be derived from two operational postulates: 1) the existence of *dynamically independent systems*; 2) the existence of *symmetric faithful states*. Both notions are crucial for the possibility of performing experiments on the system, in preventing remote instantaneous influences and in allowing calibration of apparatuses. The case of Quantum Mechanics is thoroughly analyzed. The possibility that other no-signaling theories admit a C*-algebra formulation is discussed.

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1. INTRODUCTION

In a set of recent papers [1, 2, 3] I showed how it is possible to derive the mathematical formulation of Quantum Mechanics in terms of complex Hilbert spaces and C*-algebras, starting from a small set of purely operational Postulates concerning experimental accessibility. In the present manuscript I will focus on C*-algebra, showing how a C*-algebra representation of the transformations of a physical system can be derived from two operational postulates only, concerning the existence of: 1) dynamically independent systems; 2) symmetric faithful joint states of two identical systems. Both postulates are crucial for the possibility of performing experiments, the former preventing uncontrollable remote instantaneous influences, the latter allowing calibration of experimental apparatuses.

The C*-algebra representation of the transformations is derived from the postulates via a Gelfand-Naimark-Segal (GNS) construction [4] based on the Jordan decomposition of the symmetric faithful state. The whole construction holds for finite dimensions, but it is valid also for infinite dimensions with the proviso that the Jordan decomposition exists on the Banach space of effects. The notion of *adjoint* of a transformation stems from that of faithful state, and generally depends on it, thus leading to different C*-algebra representations. On the other hand, the two postulates together imply that the linear space of "effects" of two identical independent systems is the tensor product structure of the spaces of the component systems.

A thorough analysis will show that for the case of Quantum Mechanics the adjoint is actually independent on the faithful state. However, as it will discussed in the conclusions, the C*-algebra representation of transformations is not sufficient to derive Quantum+Classical Mechanics, as for the customary operator algebras over Hilbert spaces, and in order to select this case additional postulates are needed. Possible candidates for such postulates, along with the possibility for other no-signaling theories to admit a C*-algebra representation of transformations, are discussed at the end of the paper.

2. THE POSTULATES

The general premise of the present axiomatization is the fact that one performs experiments to gather information on the *state* of an *object* physical system, and the knowledge of such state will then enable to predict the results of forthcoming experiments. Moreover, since we necessarily work with only partial *a priori* knowledge of both system and experimental apparatus, the rules for the experiment must be given in a probabilistic setting. Then, an *experiment* on an object system consists in making it interact with an apparatus, producing one of *a set of possible transformations* of the object, each one occurring with some probability. Information on the state of the object at the beginning of the experiment is gained from the knowledge of which transformation occurred, which is the "outcome" of the experiment

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signaled by the apparatus. For the above reasons we can logically *identify the experiment with a set of probabilistic transformations*.

We can now introduce the two postulates.

Postulate 1 (Independent systems) *There exist independent physical systems.*

Postulate 2 (Symmetric faithful state) *For every composite system made of two identical physical systems there exist a symmetric joint state that is both dynamically and preparationally faithful.*

3. THE STATISTICAL AND DYNAMICAL STRUCTURE

The starting point of the axiomatization is the identification **experiment** $\equiv$*set of transformations* that can occur on the object. The apparatus signals which transformation $\mathscr{A}_j$ of the set $\mathbb{A} := \{\mathscr{A}_j\}$ actually occurs. Now, since the knowledge of the state of a physical system allows us to predict the results of forthcoming experiments on the object, then it will allow us to evaluate the probability of any possible transformation in any conceivable experiment. Therefore, by definition, a **state** ω of a system is a rule providing probabilities of transformation, and $\omega(\mathscr{A})$ is the probability that the transformation $\mathscr{A}$ occurs. We clearly have the completeness $\sum_{\mathscr{A}_j\in\mathbb{A}} \omega(\mathscr{A}_j) = 1$, and assume $\omega(\mathscr{I}) = 1$ for the identical transformation $\mathscr{I}$, corresponding to adopting $\mathscr{I}$ as the free evolution (this is the *Dirac picture*, i. e. a suitable choice of the lab reference frame). In the following for a given physical system we will denote by $\mathfrak{S}$ the set of all possible states and by $\mathfrak{T}$ the set of all possible transformations.

When composing two transformations $\mathscr{A}$ and $\mathscr{B}$, the probability $p(\mathscr{B}|\mathscr{A})$ that $\mathscr{B}$ occurs conditional on the previous occurrence of $\mathscr{A}$ is given by the rule for conditional probabilities $p(\mathscr{B}|\mathscr{A}) = \omega(\mathscr{B}\circ\mathscr{A})/\omega(\mathscr{A})$. This sets a new probability rule corresponding to the notion of **conditional state** $\omega_{\mathscr{A}}$ which gives the probability that a transformation $\mathscr{B}$ occurs knowing that the transformation $\mathscr{A}$ has occurred on the object in the state ω, namely $\omega_{\mathscr{A}} \doteq \omega(\cdot\circ\mathscr{A})/\omega(\mathscr{A})$ [1] (in the following the central dot "$\cdot$" will always denote the pertinent variable). We can see that the notion of "state" itself logically implies the identification *evolution$\equiv$state-conditioning*, entailing a *linear action of transformations over states* (apart from normalization) $\mathscr{A}\omega := \omega(\cdot\circ\mathscr{A})$: this is the same concept of **operation** that we have in Quantum Mechanics, which gives the conditioning $\omega_{\mathscr{A}} = \mathscr{A}\omega/\mathscr{A}\omega(\mathscr{I})$. In other words, this is the analogous of the Schrödinger picture evolution of states in Quantum Mechanics (clearly such identification of evolution as state-conditioning also includes the deterministic case $\mathscr{U}\omega = \omega(\cdot\circ\mathscr{U})$ of transformations $\mathscr{U}$ with $\omega(\mathscr{U}) = 1\,\forall\omega\in\mathfrak{S}$—the analogous quantum channels, including unitary evolutions.

From the state-conditioning rule it follows that we can define two complementary types of equivalences for transformations: *dynamical* and *informational*. The transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **dynamically equivalent** when $\omega_{\mathscr{A}_1} = \omega_{\mathscr{A}_2}$ $\forall\omega\in\mathfrak{S}$, whereas they are **informationally equivalent** when $\omega(\mathscr{A}_1) = \omega(\mathscr{A}_2)$ $\forall\omega\in\mathfrak{S}$. The two transformations are then completely equivalent (write $\mathscr{A}_1 = \mathscr{A}_2$) when they are both dynamically and informationally equivalent, corresponding to the identity $\omega(\mathscr{B}\circ\mathscr{A}_1) = \omega(\mathscr{B}\circ\mathscr{A}_2)$, $\forall\omega\in\mathfrak{S}$, $\forall\mathscr{B}\in\mathfrak{T}$. We call **effect** the informational equivalence class of transformations[2]. In the following we will denote effects with the underlined symbols $\underline{\mathscr{A}}$, $\underline{\mathscr{B}}$, etc., or as $[\mathscr{A}]_{\text{eff}}$, and we will write $\mathscr{A}_0\in\underline{\mathscr{A}}$ meaning that "the transformation $\mathscr{A}$ belongs to the equivalence class $\underline{\mathscr{A}}$", or "$\mathscr{A}_0$ has effect $\underline{\mathscr{A}}$", or "$\mathscr{A}_0$ is informationally equivalent to $\mathscr{A}$". Since, by definition one has $\omega(\mathscr{A})\equiv\omega(\underline{\mathscr{A}})$, we will legitimately write $\omega(\underline{\mathscr{A}})$ instead of $\omega(\mathscr{A})$. Similarly, one has $\omega_{\mathscr{A}}(\mathscr{B})\equiv\omega_{\mathscr{A}}(\underline{\mathscr{B}})$, which implies that $\omega(\mathscr{B}\circ\mathscr{A}) = \omega(\underline{\mathscr{B}}\circ\mathscr{A})$, leading to the chaining rule $\underline{\mathscr{B}}\circ\mathscr{A}\in\underline{\mathscr{B}\circ\mathscr{A}}$ corresponding to the "Heisenberg picture" evolution of transformations acting on effects (notice how transformations act on effects from the right). Now, by definitions effects are linear functionals over states with range $[0,1]$, and, by duality, we have a convex structure over effects, and we will

[1] M. Ozawa noticed that the definition of conditional state needs to assume that

$$\sum_{\mathscr{B}_j\in\mathbb{B}} \omega(\mathscr{B}_j\circ\mathscr{A}) = \omega(\mathscr{A}), \qquad \forall\mathbb{B}, \forall\mathscr{A}.$$

Such assumption which seems not implicit in the present axiomatization, would correspond to a kind of *"no-signaling from the future"*. It is presently under consideration if this must be considered as an additional postulate. Notice that such assumption seems to be needed whenever a notion of conditional state is considered which involves transformations of the system. In the present context the notion of conditional state is intimately related to that of "effect" and to the action of transformations over effects.

[2] This is the same notion of "effect" introduced by Ludwig [5]

denote their convex set as $\mathfrak{P}$. An **observable** is just a complete set of effects $\mathbb{L} = \{l_i\}$ of an experiment $\mathbb{A} = \{\mathscr{A}_j\}$, namely one has $l_i = \underline{\mathscr{A}_j}\ \forall j$ (clearly, one has the completeness relation $\sum_i l_i = 1$[3]). We will call the observable $\mathbb{L} = \{l_i\}$ **informationally complete** when each effect l can be written as a real linear combination $l = \sum_i c_i(l) l_i$ of elements of $\mathbb{L}$, and when these are linearly independent we will call the informationally complete observable *minimal*.[4]

The fact that we necessarily work in the presence of partial knowledge about both object and apparatus corresponds to the possibility of incomplete specification of both states and transformations, entailing: a) the convex structure on states; b) the addition rule for **coexistent transformations**, i. e. for transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ for which $\omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \leqslant 1$, $\forall \omega \in \mathfrak{S}$ (i. e. transformations that can in principle occur in the same experiment). The addition of the two coexistent transformations is the transformation $\mathscr{S} = \mathscr{A}_1 + \mathscr{A}_2$ corresponding to the event $e = \{1,2\}$ in which the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred, but does not specify which one. Such transformation is uniquely determined by the informational and dynamical classes as $\forall \omega \in \mathfrak{S}$: $\omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2)$, $(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1\omega + \mathscr{A}_2\omega$. The composition "$\circ$" of transformations is distributive with respect to the addition "$+$". We can also define the multiplication $\lambda\mathscr{A}$ of a transformation $\mathscr{A}$ by a scalar $0 \leqslant \lambda \leqslant 1$ as the transformation dynamically equivalent to $\mathscr{A}$, but occurring with rescaled probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$. Now, since for every couple of transformations $\mathscr{A}$ and $\mathscr{B}$ the transformations $\lambda\mathscr{A}$ and $(1-\lambda)\mathscr{B}$ are coexistent for $0 \leqslant \lambda \leqslant 1$, the set of transformations also becomes a convex set. Moreover, the transformations make a *monoid* (i. e. a semigroup with identity), since the composition $\mathscr{A} \circ \mathscr{B}$ of two transformations $\mathscr{A}$ and $\mathscr{B}$ is itself a transformation, and there exists the identical transformation $\mathscr{I}$ satisfying $\mathscr{I} \circ \mathscr{A} = \mathscr{A} \circ \mathscr{I} = \mathscr{A}$ for every transformation $\mathscr{A}$. Therefore, the set of physical transformations $\mathfrak{T}$ is a convex monoid.

It is obvious that we can extend the notions of coexistence, sum and multiplication by a scalar from transformations to effects via equivalence classes. In this way also effects make a convex set. As an additional step we can extend the convex monoid of physical transformations $\mathfrak{T}$ to a real algebra $\mathfrak{T}_{\mathbb{R}}$ by taking differences of physical transformations, and multiply them by scalars $\lambda > 1$. We will call the elements of $\mathfrak{T}_{\mathbb{R}}/\mathfrak{T}$ **generalized transformations**. Likewise, we can introduce **generalized effects**, and denote their linear space as $\mathfrak{P}_{\mathbb{R}}$. On generalized effects we can introduce the norm $\|\underline{\mathscr{A}}\| := \sup_{\omega\in\mathfrak{S}} |\omega(\underline{\mathscr{A}})|$, which allows us to introduce also a norm for transformations as $\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\| = \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \sup_{\omega\in\mathfrak{S}}\ \omega(\underline{\mathscr{B}} \circ \mathscr{A})$. Closure in the respective norm topologies make $\mathfrak{P}_{\mathbb{R}}$ a real Banach space and $\mathfrak{T}_{\mathbb{R}}$ a real Banach algebra.[5]

A purely dynamical notion of **independent systems** coincides with the possibility of performing local experiments. More precisely, we say that two physical systems are *independent* if on the two systems 1 and 2 we can perform **local experiments** $\mathbb{A}^{(1)}$ and $\mathbb{A}^{(2)}$, i. e. whose transformations commute each other (i. e. $\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$, $\forall \mathscr{A}^{(1)} \in \mathbb{A}^{(1)}, \forall \mathscr{B}^{(2)} \in \mathbb{B}^{(2)}$). Notice that the above definition of independent systems is purely dynamical, in the sense that it does not contain any statistical requirement, such as the existence of factorized states. The present notion of dynamical independence is so minimal that it can be satisfied not only by the quantum tensor product, but also by the quantum direct sum [6]. Nevertheless, in Sect. 5 a dimensionality analysis will show that, in conjunction with the existence of

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[3] With a little notational abuse sometimes we identify $\underline{\mathscr{I}} \equiv 1$, i. e. the identity effect with the constant functional equal to 1.

[4] In previous literature the existence of informationally complete observable has been taken as a postulate. However, in the present context it is easy to show that it is always possible to construct a minimal informationally complete observable starting from a set of available experiments. The proof is by induction, and runs as follows. By definition there must exists a spanning set for $\mathfrak{P}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathfrak{P})$ that is contained in the convex hull $\mathfrak{P}$ of available effects. The maximal number of elements of this set that are linearly independent will constitute a *basis*, which we suppose has finite cardinality equal to $\dim(\mathfrak{P}_{\mathbb{R}})$. It remains to be shown that it is possible to have a basis with sum of elements equal to 1, and that such basis is obtained operationally starting from the available observables from which we constructed $\mathfrak{P}$.

If all observables are *uninformative* (i. e. with all constant effects $\propto \underline{\mathscr{I}}$), then $\mathfrak{P}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\underline{\mathscr{I}})$, $\underline{\mathscr{I}}$ is a minimal infocomplete observable, and the statement of the theorem is proved. Otherwise, there exists at least an observable $\mathbb{E} = \{l\}$ with $n \geqslant 2$ linearly independent effects. If this is the only observable, again the theorem is proved. Otherwise, take a new binary observable $\mathbb{B}_2 = \{x,y\}$ from the set of available ones (you can take different binary observables out of a given observable with more than two outcomes by summing up effects to yes-no observables). If $x \in \mathsf{Span}_{\mathbb{R}}(\mathbb{E})$ discard it. If $x \notin \mathsf{Span}_{\mathbb{R}}(\mathbb{E})$, then necessarily also $y \notin \mathsf{Span}_{\mathbb{R}}(\mathbb{E})$ [since if there exists coefficients λ_i such that $y = \sum_i \lambda_i l_i$, then $x = \sum_i (1-\lambda_i) l_i$]. Now, consider the observable

$$\mathbb{E}' = \left\{\tfrac{1}{2}y, \tfrac{1}{2}(l_1 + x), \tfrac{1}{2}l_2, \ldots, l_n\right\}$$

(which operationally corresponds to the random choice between the observables $\mathbb{E}$ and $\mathbb{B}_2$ with probability $\frac{1}{2}$, and with the events corresponding to x and l_1 made indistinguishable). This new observable has now $|\mathbb{E}'| = n+1$ linearly independent effects (since y is linearly independent on the l_i and one has $y = \sum_{i=1}^n l_i - x = \sum_{i=2}^n l_i + l_1 - x$). By iterating the above procedure we reach $|\mathbb{E}'| = \dim(\mathfrak{P}_{\mathbb{R}})$, and we have so realized an apparatus that measures a minimal informationally complete observable. ■

[5] An algebra of maps over a Banach space can always be made itself a Banach space, also satisfying the bound $\|\mathscr{B} \circ \mathscr{A}\| \leqslant \|\mathscr{B}\|\|\mathscr{A}\|$ defining a Banach algebra. This is true for both the real and the complex cases.

faithful states, dynamical independence agrees only with the quantum tensor product [6]. In Ref. [6] it is shown how the sole dynamical independence implies the impossibility of istantaneous signaling: the no-signaling condition is crucial for experimental control.

In the following, when dealing with more than one independent system, we will denote local transformations as ordered strings of transformations as $\mathscr{A},\mathscr{B},\mathscr{C},\ldots := \mathscr{A}^{(1)}\circ\mathscr{B}^{(2)}\circ\mathscr{C}^{(3)}\circ\ldots$. For effects one has the locality rule $([\mathscr{A}]_{\text{eff}},[\mathscr{B}_{\text{eff}}) \in [(\mathscr{A},\mathscr{B})]_{\text{eff}}$. The notion of independent systems now entails the notion of *local state*—the equivalent of partial trace in Quantum Mechanics. For two independent systems in a joint state Ω, we define the **local state** $\Omega|_1$ (and similarly $\Omega|_2$) as the probability rule $\Omega|_1(\mathscr{A}) \doteq \Omega(\mathscr{A},\mathscr{I})$ of the joint state Ω with a local transformation $\mathscr{A}$ acting only on system 1 and with all other systems untouched.

4. THE C*-ALGEBRA OF TRANSFORMATIONS

Now that we have a real algebra of generalized transformations and a linear space of generalized effects we want to introduce a positive bilinear form over them, by which we will be able to introduce a scalar product via the GNS construction [4]. The role of such bilinear form will be played by a *faithful state*.

We say that a state Φ of a bipartite system is **dynamically faithful** for system 1 when for every transformation $\mathscr{A}$ the map $\mathscr{A}\leftrightarrow(\mathscr{A},\mathscr{I})\Phi$ is one-to-one, namely $\forall\mathscr{A}\in\mathfrak{T}_{\mathbb{R}}\ (\mathscr{A},\mathscr{I})\Phi=0 \Longleftrightarrow \mathscr{A}=0$. This means that for every bipartite effect $\underline{\mathscr{B}}$ one has $\Phi(\underline{\mathscr{B}}\circ(\mathscr{A},\mathscr{I}))=0\Longleftrightarrow\mathscr{A}=0$. On the other hand, we will call a state Φ of a bipartite system **preparationally faithful** for system 1 if every joint bipartite state Ψ can be achieved by a suitable local transformation $\mathscr{T}_\Psi$ on system 1 occurring with nonzero probability, i. e. $\Psi=(\mathscr{T}_\Psi,\mathscr{I})\Phi$, with $\mathscr{T}_\Psi\in\mathfrak{T}^+$, $\mathfrak{T}^+$ denoting the positive cone generated by transformations. Clearly a bipartite state Φ that is preparationally faithful is also *locally* preparationally faithful, namely every local state ψ of system 2 can be achieved by a suitable local transformation $\mathscr{T}_\psi$ on system 1.

In Postulate 2 we also use the notion of **symmetric joint state**. This is simply defined as a joint state of two identical systems such that for any couple of effects $\underline{\mathscr{A}}$ and $\underline{\mathscr{B}}$ one has $\Phi(\underline{\mathscr{A}},\underline{\mathscr{B}})=\Phi(\underline{\mathscr{B}},\underline{\mathscr{A}})$. Clearly, for a symmetrical state the notions of dynamical and preparational faithfulness hold for both systems 1 and 2.

For a *faithful* bipartite state Φ, the **transposed transformation** $\tau_\Phi(\mathscr{A})$ of the transformation $\mathscr{A}$ is the generalized transformation which when applied to the second component system gives the same conditioned state and with the same probability as the transformation $\mathscr{A}$ operating on the first system, namely $(\mathscr{A},\mathscr{I})\Phi=(\mathscr{I},\tau_\Phi(\mathscr{A}))\Phi$ or, equivalently $\Phi(\underline{\mathscr{B}}\circ\mathscr{A},\underline{\mathscr{C}})=\Phi(\underline{\mathscr{B}},\underline{\mathscr{C}}\circ\tau_\Phi(\mathscr{A}))\ \forall\underline{\mathscr{B}},\underline{\mathscr{C}}\in\mathfrak{P}$. Clearly the transposed is unique, due to injectivity of the map $\mathscr{A}\leftrightarrow(\mathscr{A},\mathscr{I})\Phi$, and it is easy to check the axioms of transposition ($\tau_\Phi(\mathscr{A}+\mathscr{B})=\tau_\Phi(\mathscr{A})+\tau_\Phi(\mathscr{B})$, $\tau_\Phi(\tau_\Phi(\mathscr{A}))=\mathscr{A}$, $\tau_\Phi(\mathscr{A}\circ\mathscr{B})=\tau_\Phi(\mathscr{B})\circ\tau_\Phi(\mathscr{A})$) and that $\tau_\Phi(\mathscr{I})=\mathscr{I}$.

The main ingredient of a GNS construction for representing transformations would be a positive form φ over transformations based on a notion of adjoint $\mathscr{A}\to\mathscr{A}^\dagger$ by which one can construct a scalar product as $\langle\mathscr{A}|\mathscr{B}\rangle := \varphi(\mathscr{A}^\dagger\circ\mathscr{B})$ in terms of which we have $\langle\mathscr{A}|\mathscr{C}\circ\mathscr{B}\rangle=\langle\mathscr{C}^\dagger\circ\mathscr{A}|\mathscr{B}\rangle\equiv\varphi(\mathscr{A}^\dagger\circ\mathscr{C}\circ\mathscr{B})=\varphi((\mathscr{C}^\dagger\circ\mathscr{A})^\dagger\circ\mathscr{B})$.[7] We can extract from Φ a positive bilinear form over $\mathfrak{P}_{\mathbb{R}}$ (notice that the bilinear form Φ is actually defined on effects) using its **Jordan decomposition** in terms of its absolute value $|\Phi|:=\Phi_+-\Phi_-$. Indeed, the absolute value can be defined thanks to the fact that Φ is real symmetric, whence it can be diagonalized over $\mathfrak{P}_{\mathbb{R}}$ in the finite dimensional case. Upon denoting by $\mathscr{P}_\pm$ the orthogonal projectors over the linear space corresponding to positive and negative eigenvalues, respectively, [8] one has $|\Phi|(\underline{\mathscr{A}},\underline{\mathscr{B}})=\Phi(\varsigma_\Phi(\underline{\mathscr{A}}),\underline{\mathscr{B}})$, where $\varsigma_\Phi(\underline{\mathscr{A}}):=(\mathscr{P}_+-\mathscr{P}_-)(\underline{\mathscr{A}})$. The map ς_Φ is an involution, namely $\varsigma_\Phi^2=\mathscr{I}$. The fact that the state is also preparationally faithful implies that the bilinear form is *strictly* positive [1] (namely $|\Phi|(\underline{\mathscr{A}},\underline{\mathscr{A}})=0$ implies that $\underline{\mathscr{A}}=0$). The involution ς_Φ over $\mathfrak{P}_{\mathbb{R}}$ corresponds to a generalized transformation $\mathscr{Z}_\Phi\in\mathfrak{T}_{\mathbb{R}}$ defined as $\underline{\mathscr{A}}\circ\mathscr{Z}_\Phi:=\varsigma_\Phi(\underline{\mathscr{A}})$, whence it can be extended to generalized transformations $\mathfrak{T}_{\mathbb{R}}$ via $\underline{\mathscr{B}}\circ\varsigma_\Phi(\mathscr{A})=\varsigma_\Phi(\varsigma_\Phi(\underline{\mathscr{B}})\circ\mathscr{A})$, corresponding to $\varsigma_\Phi(\mathscr{A})=\mathscr{Z}_\Phi\circ\mathscr{A}\circ\mathscr{Z}_\Phi$. Since $\mathscr{Z}_\Phi^2=\mathscr{I}$ the extension of ς_Φ to $\mathfrak{T}_{\mathbb{R}}$ is composition-preserving, i. e. $\varsigma_\Phi(\mathscr{B}\circ\mathscr{A})=\varsigma_\Phi(\mathscr{B})\circ\varsigma_\Phi(\mathscr{A})$.

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[6] As shown in Refs. [1, 6] the tensor product can be derived from the additional Postulate stating the *local observability principle*.

[7] It is not easy to devise a positive form over generalized transformations $\mathfrak{T}_{\mathbb{R}}$ such that the transposition plays the role of the adjoint on a real Hilbert space. Indeed, if we take φ as the local state of a symmetric faithful state $\varphi=\Phi|_2\equiv\Phi|_1$ we have $\varphi(\tau_\Phi(\mathscr{A})\circ\mathscr{B})=\Phi(\tau_\Phi(\mathscr{A}),\tau_\Phi(\mathscr{B}))$, but the fact that Φ is positive over the convex set $\mathfrak{T}$ of physical transformations doesn't guarantee that its extension to generalized transformations $\mathfrak{T}_{\mathbb{R}}$ is still positive.

[8] The existence of the orthogonal space decomposition corresponding to positive and negative eigenvalues is guaranteed for finite dimensions. For infinite dimensions Φ is just a symmetric form over a real Banach space—the space $\mathfrak{P}_{\mathbb{R}}$ of generalized effects—and the existence of such decomposition needs to be proven.

The explicit form of $\mathscr{Z}_\Phi$ can be obtained in terms of the basis $\{f_j\}$ for $\mathfrak{P}_\mathbb{R}$ reducing the bilinear symmetric form Φ over $\mathfrak{P}_\mathbb{R}$ to the canonical form

$$\Phi(f_i, f_j) = s_i \delta_{ij}, \tag{1}$$

where $s_i = \pm 1$ is the signature of the eigenvector f_i. Then one has

$$\varsigma_\Phi(\underline{\mathscr{A}}) = \underline{\mathscr{A}} \circ \mathscr{Z}_\Phi = \sum_j \Phi(f_j, \underline{\mathscr{A}}) f_j. \tag{2}$$

One can see that $\tau_\Phi \varsigma_\Phi = \varsigma_\Phi \tau_\Phi$. In fact, due to the symmetry of Φ, $\tau_\Phi(\mathscr{Z}_\Phi) = \mathscr{Z}_\Phi$, since for any couple of elements f_k, f_l of the basis

$$\Phi(f_k \circ \tau_\Phi(\mathscr{Z}_\Phi), f_l) = \Phi(f_k, f_l \circ \mathscr{Z}_\Phi) = \Phi(f_l \circ \mathscr{Z}_\Phi, f_k) = \delta_{lk} = \Phi(f_k \circ \mathscr{Z}_\Phi, f_l). \tag{3}$$

whence

$$\begin{aligned} \tau_\Phi(\varsigma_\Phi(\mathscr{A})) =& \tau_\Phi(\mathscr{Z}_\Phi \circ \mathscr{A} \circ \mathscr{Z}_\Phi) = \tau_\Phi(\mathscr{Z}_\Phi) \circ \tau_\Phi(\mathscr{A}) \circ \tau_\Phi(\mathscr{Z}_\Phi) \\ =& \mathscr{Z}_\Phi \circ \tau_\Phi(\mathscr{A}) \circ \mathscr{Z}_\Phi = \varsigma_\Phi(\tau_\Phi(\mathscr{A})). \end{aligned} \tag{4}$$

We now define the **adjoint map** $\mathrm{ad}_\Phi := \varsigma_\Phi \tau_\Phi = \tau_\Phi \varsigma_\Phi$. Here in the following we will also temporarily use the more compact notation $\mathscr{A}^\dagger := \mathrm{ad}_\Phi(\mathscr{A})$, keeping in mind that the definition of the adjoint generally depends on the faithful state Φ with respect to which it is defined. Since ς_Φ is composition preserving whereas τ_Φ is a transposition, one has $(\mathscr{B} \circ \mathscr{A})^\dagger = \mathscr{A}^\dagger \circ \mathscr{B}^\dagger$. Moreover, for $\varphi = \Phi|_1$ we have that $\varphi(\mathscr{A}^\dagger \circ \mathscr{B}) = |\Phi|(\tau_\Phi(\mathscr{A}), \tau_\Phi(\mathscr{B}))$ is a positive bilinear form over transformations (strictly positive over effects, i. e. $|\Phi|(\underline{\mathscr{A}}, \underline{\mathscr{A}}) = 0 \Rightarrow \underline{\mathscr{A}} = 0$), and can be used to define a scalar product over transformations as follows

$$_\Phi\langle \mathscr{A} | \mathscr{B} \rangle_\Phi := \varphi(\mathscr{A}^\dagger \circ \mathscr{B}) = \Phi(\varsigma_\Phi \tau_\Phi(\mathscr{A}), \tau_\Phi(\mathscr{B})). \tag{5}$$

We can then verify that $\mathscr{A}^\dagger := \varsigma_\Phi \tau_\Phi(\mathscr{A})$ works as an adjoint for such scalar product, namely one has $_\Phi\langle \mathscr{C}^\dagger \circ \mathscr{A} | \mathscr{B} \rangle_\Phi = {_\Phi\langle} \mathscr{A} | \mathscr{C} \circ \mathscr{B} \rangle_\Phi$. In this way ς_Φ is identified as the **complex conjugation**, and as usual the adjoint $\mathscr{A}^\dagger := \varsigma_\Phi \tau_\Phi(\mathscr{A}) = \tau_\Phi \varsigma_\Phi(\mathscr{A})$ is the composition of the transposition with the complex conjugation. Now, by taking complex linear combinations of generalized transformations and defining $\varsigma_\Phi(c\mathscr{A}) = c^* \varsigma_\Phi(\mathscr{A})$ for $c \in \mathbb{C}$, we can extend the adjoint to complex linear combinations of generalized transformations, whose linear space will be denoted by $\mathcal{A} \equiv \mathfrak{T}_\mathbb{C}$, which is a complex algebra. On the other hand, we can trivially extend the real linear space of generalized effects $\mathfrak{P}_\mathbb{R}$ to a complex linear space $\mathfrak{P}_\mathbb{C}$ by taking complex linear combinations of generalized effects. The remaining setting up of the C*-algebra representation of $\mathcal{A}$ is just standard GNS construction, starting from the scalar product between transformations in Eq. (5). Symmetry and positivity imply the bounding [1] $_\Phi\langle \mathscr{A} | \mathscr{B} \rangle_\Phi \leqslant \|\mathscr{A}\|_\Phi \|\mathscr{B}\|_\Phi$, where we introduced the norm induced by the scalar product $\|\mathscr{A}\|^2_\Phi \doteq {_\Phi\langle} \mathscr{A} | \mathscr{A} \rangle_\Phi$. By taking the equivalence classes $\mathcal{A}/\mathcal{I}$ with respect to the zero-norm elements $\mathcal{I} \subseteq \mathcal{A}$ we thus obtain a complex pre-Hilbert space equipped with a symmetric scalar product, and, since the scalar product is strictly positive over generalized effects, the elements of $\mathcal{A}/\mathcal{I}$ are indeed the generalized effects, i. e. $\mathcal{A}/\mathcal{I} \simeq \mathfrak{P}_\mathbb{C}$ as linear spaces. Being endowed with the scalar product (5) $\mathcal{A}/\mathcal{I}$ becomes a pre-Hilbert space, whose completion $\mathsf{H}_\Phi := \overline{\mathcal{A}/\mathcal{I}}$ under the norm induced by the scalar product is then a Hilbert space. In the following we will conveniently denote the equivalence class of transformations containing $\mathscr{A}$ in $\mathcal{A}/\mathcal{I}$ by the Dirac vector itself $|\mathscr{A}\rangle_\Phi \in \mathsf{H}_\Phi$. From the bounding for the scalar product it follows that the set $\mathcal{I} \subseteq \mathcal{A}$ of zero norm elements $\mathscr{X} \in \mathcal{A}$ is a left ideal (i. e. $\mathscr{X} \in \mathcal{I}$, $\mathscr{A} \in \mathcal{A}$ implies $\mathscr{A} \circ \mathscr{X} \in \mathcal{I}$), whence using our scalar product defined as in Eq. (5) we can represent elements of $\mathcal{A}$ (i. e. generalized complex transformations, since $\mathcal{A} \equiv \mathfrak{T}_\mathbb{C}$) as operators over the pre-Hilbert space of effects $\mathfrak{P}_\mathbb{C}$. The product in $\mathcal{A}$ defines the action of $\mathcal{A}$ on the vectors in $\mathcal{A}/\mathcal{I}$, by associating to each element $\mathscr{A} \in \mathcal{A}$ the linear operator $\pi_\Phi(\mathscr{A})$ defined on the dense domain $\mathcal{A}/\mathcal{I} \subseteq \mathsf{H}_\Phi$ as $\pi_\Phi(\mathscr{A}) |\mathscr{B}\rangle_\Phi \doteq |\mathscr{A} \circ \mathscr{B}\rangle_\Phi$. The fact that $\mathcal{A}$ is a Banach algebra[9] also implies that the domain of definition of $\pi_\Phi(\mathscr{A})$ can be easily extended to the whole H_Φ by continuity. Being now an operator algebra over a complex Hilbert space, $\mathcal{A}$ becomes a C*-algebra. We just need to introduce the norm on transformations as the respective operator norm over H_Φ, namely $\|\mathscr{A}\|_\Phi := \sup_{v \in \mathsf{H}_\Phi, \|v\|_\Phi \leqslant 1} \|\mathscr{A} v\|_\Phi$, and completion of $\mathcal{A}$ under the norm topology will give a C*-algebra

[9] Indeed norms introduced in Sect. 3 can be extended to the respective complex linear spaces, and the norm completion makes $\mathcal{A}$ also a complex Banach algebra, as explained in the footnote 5.

(i. e. a complex Banach algebra satisfying the identity $\|\mathscr{A}^\dagger \circ \mathscr{A}\| = \|\mathscr{A}\|^2$), as it can be easily proved by standard techniques [1].

I want to emphasize that even though $\mathsf{H}_\Phi \simeq \mathfrak{P}_\mathbb{C}$ as linear spaces, the elements $|\mathscr{A}\rangle_\Phi \in \mathsf{H}_\Phi$ should be regarded as element of the dual space of $\mathfrak{P}_\mathbb{C}$, in the sense that the action of transformations over vectors $|\mathscr{A}\rangle_\Phi \in \mathsf{H}_\Phi$ is from the left—as in the Schrődinger picture—instead of being from the right—as in the Heisenberg picture, e. g. $\pi_\Phi(\mathscr{C})|\mathscr{A}\rangle_\Phi = |\mathscr{C}\circ\mathscr{A}\rangle_\Phi =$, or ${}_\Phi\langle\mathscr{A}|\pi_\Phi(\mathscr{C}) = {}_\Phi\langle\mathscr{C}^\dagger\circ\mathscr{A}|$, as it follows from the identity $\langle\mathscr{B}|\pi_\Phi(\mathscr{C})|\mathscr{A}\rangle_\Phi = \langle\mathscr{B}|\mathscr{C}\circ\mathscr{A}\rangle_\Phi = \langle\mathscr{C}^\dagger\circ\mathscr{B}|\underline{\mathscr{A}}\rangle_\Phi$. The Schrődinger picture is obtained thanks to the transposition in the definition of the scalar product ${}_\Phi\langle\mathscr{B}|\mathscr{A}\rangle_\Phi = |\Phi|(\tau_\Phi(\mathscr{B}),\tau_\Phi(\mathscr{A}))$.

From the definition of the scalar product, and using the fact that the state Φ is also preparationally faithful according to Postulate 2, the Born rule can be written in the GNS representation as $\omega(\underline{\mathscr{A}}) = {}_\Phi\langle\mathscr{A}^\dagger|\rho\rangle_\Phi$, with representation of state $\rho = \tau_\Phi(\underline{\mathscr{T}_\omega})/\Phi(\underline{\mathscr{T}}_\omega,\mathscr{I})$ [1], $\mathscr{T}_\omega$ denoting the transformation on system 2 corresponding to the local state ω on system 1, namely $\omega \propto \Phi(\cdot,\mathscr{T}_\omega)$. Then, the representation of transformations is

$$\omega(\underline{\mathscr{B}}\circ\mathscr{A}) = {}_\Phi\langle\mathscr{B}^\dagger|\mathscr{A}|\rho\rangle_\Phi := {}_\Phi\langle\mathscr{B}^\dagger|\mathscr{A}\circ\rho\rangle_\Phi. \tag{6}$$

4.1. Connecting two faithful states

Suppose that Ω is a symmetric state which is faithful both preparationally and dynamically, and that Φ is another such kind of state. Then, there must exists an invertible generalized transformation $\mathscr{F}$ in the positive cone $\mathfrak{T}^+$ generated by physical transformations, such that

$$\Phi = (\mathscr{F},\mathscr{I})\Omega. \tag{7}$$

In fact, since Ω is preparationally faithful, there must exists a local physical transformation which transforms Ω into any state with some probability. On the other hand, since Ω is dynamically faithful, in order to have also Φ so, the correspondence between any other joint state and a local map applied to Φ must be one-to-one, which is true iff $\mathscr{F}$ is invertible. If the map $\mathscr{F}^{-1}$ is itself in the positive cone $\mathfrak{T}^+$ generated by physical transformations, then the state is also preparationally faithful, and viceversa. Indeed, any pure joint state Σ must be written as $\Sigma = (\mathscr{S},\mathscr{I})\Omega$ with $\mathscr{S}\in\mathfrak{T}^+$. Therefore Σ can also be obtained probabilistically from Φ as $(\tilde{\mathscr{S}},\mathscr{I})\Phi$ using a transformation $\tilde{\mathscr{S}} \propto \mathscr{F}\mathscr{S}\mathscr{F}^{-1}\in\mathfrak{T}^+$ belonging to the convex cone $\mathfrak{T}^+$ generated by physical transformations. Finally, as regards symmetry, the state Φ is symmetric iff $\tau_\Omega(\mathscr{F}) = \mathscr{F}$, since

$$\begin{aligned}\Phi(\mathscr{A},\mathscr{B}) =& \Omega(\mathscr{A}\circ\mathscr{F},\mathscr{B}) = \Omega(\mathscr{B},\mathscr{A}\circ\mathscr{F}) = \Omega(\mathscr{B}\circ\tau_\Omega(\mathscr{F}),\mathscr{A}),\\ \Phi(\mathscr{A},\mathscr{B}) =& \Phi(\mathscr{B},\mathscr{A}) = \Omega(\mathscr{B}\circ\mathscr{F},\mathscr{A}), \quad \forall\mathscr{B},\mathscr{A}\in\mathfrak{T}\end{aligned} \tag{8}$$

and using preparational faithfulness of Ω one can see that the above identity holds true iff $\tau_\Omega(\mathscr{F}) = \mathscr{F}$ (we remind that two transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are equal iff $\omega(\mathscr{B}\circ\mathscr{A}_1) = \omega(\mathscr{B}\circ\mathscr{A}_2)$ $\forall\omega\in\mathfrak{S}$ and $\forall\mathscr{B}\in\mathfrak{T}$). Notice now that $\tau_\Omega(\mathscr{F}^{-1}) = \mathscr{F}^{-1}$, since $\mathscr{I} = \tau_\Omega(\mathscr{F}^{-1})\circ\tau_\Omega(\mathscr{F}) = \tau_\Omega(\mathscr{F}^{-1})\circ\mathscr{F}$.

The transposed with respect to Φ is obtained as follows

$$(\mathscr{A},\mathscr{I})(\mathscr{F},\mathscr{I})\Omega = (\mathscr{I},\tau_\Phi(\mathscr{A}))(\mathscr{F},\mathscr{I})\Omega = (\mathscr{I},\tau_\Phi(\mathscr{A})\circ\tau_\Omega(\mathscr{F}))\Omega \tag{9}$$

namely $\tau_\Phi(\mathscr{A})\circ\tau_\Omega(\mathscr{F}) = \tau_\Omega(\mathscr{F})\circ\tau_\Omega(\mathscr{A})$, which means that

$$\tau_\Phi(\mathscr{A}) = \tau_\Omega(\mathscr{F}^{-1}\circ\mathscr{A}\circ\mathscr{F}) = \mathscr{F}\circ\tau_\Omega(\mathscr{A})\circ\mathscr{F}^{-1} \tag{10}$$

The canonical basis of eigenvectors $\{f_j\}$ of the bilinear form Φ must satisfy the identities

$$s_j\delta_{ij} = \Phi(f_i,f_j), \quad \delta_{ij} = \Phi(\varsigma_\Phi(f_i),f_j) = |\Phi|(f_i,f_j), \tag{11}$$

and upon multiplying by f_j and summing over the index j one obtains $f_j\circ\mathscr{Z}_\Phi = \sum_j\Phi(f_i,f_j)f_j$, and since $\{f_i\}$ is a basis for $\mathfrak{P}_\mathbb{R}$, one as the identity

$$\underline{\mathscr{A}}\circ\mathscr{Z}_\Phi = \sum_j\Phi(\underline{\mathscr{A}},f_j)f_j, \quad \forall\underline{\mathscr{A}}\in\mathfrak{P}_\mathbb{R}. \tag{12}$$

For any couple of elements of the complete basis $\{f_j\}$ for $\mathfrak{P}_\mathbb{R}$ one has

$$\delta_{ij} = |\Phi|(f_i,f_j) = \Phi(f_i\circ\mathscr{Z}_\Phi,f_j) = \Omega(f_i\circ\mathscr{Z}_\Phi\circ\mathscr{F},f_j), \tag{13}$$

and since $\{f_i\}$ is a basis for $\mathfrak{P}_{\mathbb{R}}$, this corresponds to the identity

$$\mathscr{Z}_\Phi \circ \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}} = \mathscr{I}, \tag{14}$$

where

$$\underline{\mathscr{A}} \circ \mathscr{Z}_{\Omega,\mathbf{f}} := \sum_j \Omega(\underline{\mathscr{A}}, f_j) f_j, \quad \forall \underline{\mathscr{A}} \in \mathfrak{P}_{\mathbb{R}}. \tag{15}$$

The definition of $\mathscr{Z}_{\Omega,\mathbf{f}}$ generalizes that of $\mathscr{Z}_\Phi$ in specifying the basis $\mathbf{f} := \{f_j\}$ which is generally non canonical for Ω. For $\mathbf{o} := \{o_j\}$ canonical for Ω one has simply $\mathscr{Z}_\Omega \equiv \mathscr{Z}_{\Omega,\mathbf{o}}$. Upon multiplying by f_j and summing over the index j in Eq. (44) we obtain

$$s_i f_i = \sum_j \Phi(f_i, f_j) f_j = \sum_j \Omega(f_i \circ \mathscr{F}, f_j) f_j = f_i \circ \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}}. \tag{16}$$

This corresponds to

$$\mathscr{Z}_\Phi = \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}}, \tag{17}$$

which, in conjunction with Eq. (14), is a restatement of the involutive nature of $\mathscr{Z}_\Phi$, i. e. $\mathscr{Z}_\Phi \circ \mathscr{Z}_\Phi = \mathscr{I}$, corresponding also to the identities

$$\mathscr{Z}_{\Omega,\mathbf{f}} \circ \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}} = \mathscr{F}^{-1}, \quad \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}} \circ \mathscr{F} = \mathscr{Z}_{\Omega,\mathbf{f}}^{-1}. \tag{18}$$

Therefore, one also has

$$\mathscr{Z}_\Phi = \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}} = \mathscr{Z}_{\Omega,\mathbf{f}}^{-1} \circ \mathscr{F}^{-1}. \tag{19}$$

The complex conjugation obeys the symmetry $\tau_\Phi(\mathscr{Z}_\Phi) = \mathscr{Z}_\Phi$ which is needed for a proper definition of the adjoint. Indeed

$$\tau_\Phi(\mathscr{Z}_\Phi) = \mathscr{F} \circ \tau_\Omega(\mathscr{Z}_\Phi) \circ \mathscr{F}^{-1} = \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}} \circ \mathscr{F} \circ \mathscr{F}^{-1} = \mathscr{F} \circ \mathscr{Z}_{\Omega,\mathbf{f}} = \mathscr{Z}_\Phi. \tag{20}$$

One has $\tau_\Omega(\mathscr{Z}_{\Omega,\mathbf{f}}) = \mathscr{Z}_{\Omega,\mathbf{f}}$, since

$$\begin{aligned}\Omega(f_i \circ \mathscr{Z}_{\Omega,\mathbf{f}}, f_j) =& \sum_k \Omega(f_i, f_k)\Omega(f_k, f_j) = \sum_k \Omega(f_j, f_k)\Omega(f_k, f_i) \\ =& \Omega(f_j \circ \mathscr{Z}_{\Omega,\mathbf{f}}, f_i) = \Omega(f_i, f_j \circ \mathscr{Z}_{\Omega,\mathbf{f}})\end{aligned} \tag{21}$$

We now evaluate the adjoint

$$\mathrm{ad}_\Phi(\mathscr{A}) := \varsigma_\Phi \tau_\Phi(\mathscr{A}) = \mathscr{Z}_\Phi \circ \mathscr{F} \circ \tau_\Omega(\mathscr{A}) \circ \mathscr{F}^{-1} \circ \mathscr{Z}_\Phi = \mathscr{Z}_{\Omega,\mathbf{f}}^{-1} \circ \tau_\Omega(\mathscr{A}) \circ \mathscr{Z}_{\Omega,\mathbf{f}}, \tag{22}$$

and one has

$$\mathrm{ad}_\Phi \equiv \mathrm{ad}_\Omega := (\cdot)^\dagger \quad \Leftrightarrow \quad \mathscr{Z}_{\Omega,\mathbf{f}} \equiv \mathscr{Z}_\Omega, \tag{23}$$

namely if $\mathscr{Z}_\Omega \equiv \mathscr{F}^{-1} \circ \mathscr{Z}_\Phi$. In such case we will also have

$$\mathscr{F}^{-1} = \varsigma_\Omega(\mathscr{F}) = \mathscr{F}^\dagger. \tag{24}$$

5. DYNAMICAL INDEPENDENCE AND TENSOR PRODUCT

As already mentioned, our notion of dynamical independence—i. e. the possibility of performing local experiments—can be satisfied not only by the quantum tensor product, but also by the quantum direct sum. This is shown in detail in Ref. [6]. Here I will show how Postulate 2—the existence of dynamically and preparationally faithful states—in conjunction with dynamical independence, leads to the right dimension for the convex set of states of two independent identical systems according to the tensor product rule.

The state-effect duality leads to the identity $\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$,[10] (we remind that one dimension is blocked by state normalization). Then, the existence of a preparationally and dynamically faithful state guarantees that generalized transformations and generalized joint states are isomorphic as linear spaces, whence $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$, $\mathfrak{S}^{\times 2}$ denoting the set of bipartite states of two identical systems, each with set of states $\mathfrak{S}$. Finally, the GNS construction represents generalized transformations as operators over the Hilbert space of generalized effects, whence $\dim(\mathfrak{T}) = \dim(\mathfrak{P})^2$, from which it follows that $\dim(\mathfrak{S}^{\times 2}) + 1 = (\dim(\mathfrak{S}) + 1)^2$. Therefore one has $\dim(\mathfrak{P}^{\times 2}) = (\dim \mathfrak{P})^2$, and $\dim_{\mathbb{C}}(\mathfrak{P}_{\mathbb{C}}^{\times 2}) = (\dim \mathfrak{P}_{\mathbb{C}})^2$ (since $\dim_{\mathbb{C}} \mathfrak{P}_{\mathbb{C}} = \frac{1}{2} \dim \mathfrak{P}_{\mathbb{C}} = \dim \mathfrak{P}_{\mathbb{R}}$), whence $\mathfrak{P}_{\mathbb{R}}^{\times 2} \equiv \mathfrak{P}_{\mathbb{R}}^{\otimes 2}$ and $\mathfrak{P}_{\mathbb{C}}^{\times 2} \equiv \mathfrak{P}_{\mathbb{C}}^{\otimes 2}$. The last identities hold in Quantum Mechanics, as a consequence of the tensor product of complex Hilbert spaces.

[10] For convex sets $\mathfrak{C}$, one has $\dim(\mathfrak{C}) := \dim \mathsf{Span}(\mathfrak{C})$, where $\dim \equiv \dim_{\mathbb{R}}$ (if not otherwise stated, the convex sets are always considered real).

6. THE QUANTUM C*-ALGEBRA OF TRANSFORMATIONS

In the following, for given fixed orthonormal basis $\{|j\rangle\}$ for H we will denote by $A^* = \sum_{ij} A^*_{ij}|i\rangle\langle j|$ the operator corresponding to the complex conjugated matrix of $A = \sum_{ij} A_{ij}|i\rangle\langle j|$, and consistently $A^t = (A^*)^\dagger$ will denote the transposed-matrix operator. With the double ket we denote bipartite vectors $|\Psi\rangle\!\rangle \in \mathcal{H}\otimes\mathcal{H}$, which, keeping the basis $\{|j\rangle\}$ as fixed, are in one-to-one correspondence with matrices as $|\Psi\rangle\!\rangle = \sum_{ij}\Psi_{ij}|i\rangle\otimes|j\rangle$. We will denote the generalized transformation and the corresponding quantum linear map by the same letter, and we will do so also for state and its corresponding quantum density operator. Moreover, we will write composition of quantum maps as $\mathscr{B}\mathscr{A}$ as usual, instead of using the operational notation $\mathscr{B}\circ\mathscr{A}$. In Quantum Mechanics physical transformations correspond to quantum operations (i. e. trace non-increasing completely positive (CP) maps), effects correspond to positive contractions, generalized transformations $\mathfrak{T}_\mathbb{R}$ to differences of CP maps, and generalized effects $\mathfrak{P}_\mathbb{R}$ to selfadjoint operators. In the following we will denote by $P_\mathscr{A}$ the positive operator describing the effect of the quantum operation $\mathscr{A}$. For example, we will write

$$\rho(\mathscr{A}) = \mathrm{Tr}[\mathscr{A}(\rho)] = \mathrm{Tr}\left[\sum_n A_n\rho A_n^\dagger\right] = \mathrm{Tr}[\rho P_\mathscr{A}], \qquad P_\mathscr{A} := \sum_n A_n^\dagger A_n. \tag{25}$$

We will also use the notation $\mathscr{A}^\dagger = \sum_n A_n^\dagger \cdot A_n$ for the usual adjoint map of $\mathscr{A} = \sum_n A_n\cdot A_n^\dagger$, and $\mathscr{A}^t = \sum_n A_n{}^t\cdot A_n^*$ for the transposed map.

I will now construct explicitly the C*-algebra $\mathfrak{T}_\mathbb{C}$ of c-generalized transformations for a general faithful symmetric quantum state Φ. I first consider the case of the canonical maximally entangled state Ω, and then analyze the general case of faithful symmetric state.

6.1. The maximally entangled state of a qudit

The canonical maximally entangled state of a qudit

$$\Omega = d^{-1}|I\rangle\!\rangle\langle\!\langle I|, \tag{26}$$

is faithful, both dynamically and preparationally. The fact that it is dynamically faithful is just the Choi-Jamiolowski representation of CP maps. On the other hand, any pure joint state $d^{-\frac{1}{2}}|S\rangle\!\rangle$ can be written as $(S\otimes I)d^{-\frac{1}{2}}|I\rangle\!\rangle$ with $d^{-1}\,\mathrm{Tr}[S^\dagger S]=1$, $\mathscr{S}\propto S\cdot S^\dagger$ quantum operation (i. e. $\mathscr{S}\in\mathfrak{T}^+$), whence Ω is preparationally faithful. The state Ω is also symmetric, since for any couple of generalized effects one has

$$\Omega(\mathscr{A},\mathscr{B}) = \mathrm{Tr}[\mathscr{A}\otimes\mathscr{B}(\Omega)] = \tfrac{1}{d}\,\mathrm{Tr}[P_\mathscr{A}P_\mathscr{B}^t] = \tfrac{1}{d}\,\mathrm{Tr}[P_\mathscr{B}P_\mathscr{A}^t] = \Omega(\mathscr{B},\mathscr{A}). \tag{27}$$

The transposition τ_Ω is just the customary transposition $\tau_\Omega \equiv (\cdot)^t$ with respect to any fixed basis $\{|i\rangle\}$ such that Ω has all probability amplitudes equal to $d^{-\frac{1}{2}}$. Indeed, it is easy to check that

$$(\mathscr{A}\otimes\mathscr{I})(|I\rangle\!\rangle\langle\!\langle I|) = (\mathscr{I}\otimes\mathscr{A}^t)(|I\rangle\!\rangle\langle\!\langle I|). \tag{28}$$

In order to construct an eigenbasis for the Jordan form, consider the following selfadjoint operators

$$X_{kl} = \tfrac{1}{\sqrt{2}}(|k\rangle\langle l| + |l\rangle\langle k|), \quad Y_{kl} = \tfrac{i}{\sqrt{2}}(|k\rangle\langle l| - |l\rangle\langle k|), \quad k<l, \qquad Z_l = |l\rangle\langle l|. \tag{29}$$

One has

$$\mathrm{Tr}[X_{kl}X_{k'l'}] = \delta_{kl'}\delta_{lk'} + \delta_{kk'}\delta_{ll'} \equiv \delta_{kk'}\delta_{ll'}, \tag{30}$$

since for $k=l'>k'$ one has $k'=l>k$. Similarly we have $\mathrm{Tr}[Y_{kl}Y_{k'l'}] = \delta_{kk'}\delta_{ll'}$, and $\mathrm{Tr}[Z_kZ_{k'}] = \delta_{kk'}$, and, moreover

$$\mathrm{Tr}[X_{kl}Y_{k'l'}] = \mathrm{Tr}[Z_lY_{k'l'}] = \mathrm{Tr}[Z_lX_{k'l'}] = 0. \tag{31}$$

Therefore, the following is a canonical basis for the Jordan form of Φ

$$\begin{aligned}[C_j] = \big[&Z_0, Z_1, \ldots, Z_{d-1}, X_{01}, X_{02}, \ldots, X_{0,d-1},\\ &X_{12}, X_{13}, \ldots, X_{1,d-1}, \ldots X_{d-2,d-1}, Y_{01}, \ldots, Y_{d,d-1}\big],\end{aligned} \tag{32}$$

with Jordan form

$$\Omega(C_i, C_j) = \mathrm{Tr}[C_i C_j^*] = \delta_{ij} s_j, \tag{33}$$

Y_{kl} ($0 \leqslant k < l \leqslant d-1$) spanning the eigenspace with negative eigenvalue of the symmetric form Ω. It follows that the transformation ς_Ω corresponds to the complex conjugation $\varsigma_\Omega \equiv (\cdot)^*$ with respect to the same fixed orthonormal basis $\{|i\rangle\}$ used for transposition. We can construct the Kraus form for the corresponding generalized transformation $\mathscr{Z}_\Omega$, passing through the construction of the corresponding Choi-Jamiolowski operator

$$\mathscr{Z}_\Omega \otimes \mathscr{I}(|I\rangle\!\rangle\langle\!\langle I|) = \mathscr{Z}_\Omega \otimes \mathscr{I}\left(\sum_j C_j^* \otimes C_j\right) = \sum_j C_j \otimes C_j = \sum_j C_j^\dagger \otimes C_j = E, \tag{34}$$

which is just the unitary swap operator E, with eigenvectors

$$E|C_j\rangle\!\rangle = |C_j^*\rangle\!\rangle = s_j|C_j\rangle\!\rangle, \tag{35}$$

corresponding to the Kraus form for the generalized transformation $\mathscr{Z}$

$$\mathscr{Z} = \sum_j s_j C_j \cdot C_j. \tag{36}$$

The GNS representation of transformations over effects is provided by the following scalar product

$${}_\Omega\langle \mathscr{A}|\mathscr{B}\rangle_\Omega := \tfrac{1}{d}\langle\!\langle I|\check{A}^\dagger \check{B}|I\rangle\!\rangle, \tag{37}$$

where corresponding to the map $\mathscr{A} = \sum_n A_n \cdot A_n^\dagger$ we define the operator $\check{A} := \sum_n A_n \otimes A_n^*$ such that $\check{A}|X\rangle\!\rangle = |\mathscr{A}(X)\rangle\!\rangle$. Indeed, we can check the identities

$${}_\Omega\langle \mathscr{A}|\mathscr{B}\rangle_\Omega := \Omega(\mathscr{A}^\dagger, \mathscr{B}') = \tfrac{1}{d}\,\mathrm{Tr}\left[\sum_m A_m A_m^\dagger \sum_n B_n B_n^\dagger\right] = \tfrac{1}{d}\,\mathrm{Tr}[P_{\mathscr{A}^\dagger} P_{\mathscr{B}^\dagger}], \tag{38}$$

$${}_\Omega\langle \mathscr{A}|\mathscr{C} \circ \mathscr{B}\rangle_\Omega := \Phi(\mathscr{A}^\dagger, \mathscr{B}' \circ \mathscr{C}') = \tfrac{1}{d}\langle\!\langle P_{\mathscr{A}^\dagger}|\check{C}|P_{\mathscr{B}^\dagger}\rangle\!\rangle = \tfrac{1}{d}\,\mathrm{Tr}[P_{\mathscr{A}^\dagger}\mathscr{C}(P_{\mathscr{B}^\dagger})] \tag{39}$$

Explicitely, the GNS representation of transformation over effects is

$$|\mathscr{A}\rangle = \check{A}|I\rangle\!\rangle = |\mathscr{A}(I)\rangle\!\rangle = |P_{\mathscr{A}^\dagger}\rangle\!\rangle, \quad \mathscr{B}|\mathscr{A}\rangle = \check{B}|\mathscr{A}(I)\rangle\!\rangle = |\mathscr{B}\mathscr{A}(I)\rangle\!\rangle = |\mathscr{B}(P_{\mathscr{A}^\dagger})\rangle\!\rangle. \tag{40}$$

For qubits the canonical Jordan basis, will be given by the set of four Pauli matrices $\sigma_0 \equiv I, \sigma_x, \sigma_y, \sigma_z$ normalized as $C_j = \frac{1}{\sqrt{2}}\sigma_j$, corresponding to the Jordan form

$$\Omega(C_i, C_j) \doteq \tfrac{1}{2}\,\mathrm{Tr}[\sigma_i \sigma_j^*] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} := \delta_{ij} s_j. \tag{41}$$

Here σ_y spans the eigenspace with negative eigenvalue of Ω.

6.2. General faithful state

According to subsection 4.1 the general form of a joint faithful state of two identical quantum systems with finite dimensional Hilbert space H can be always recast in the following way

$$\Phi = \tfrac{1}{d}\sum_l |F_l\rangle\!\rangle\langle\!\langle F_l| = (\mathscr{F} \otimes \mathscr{I})\Omega \tag{42}$$

with Ω given in Eq. (26), and $\mathscr{F} = \sum_l F_l \cdot F_l^\dagger$ invertible CP map (not necessarily trace-non-increasing), and with $\mathscr{F}^{-1}$ also CP (normalization corresponds to $\mathrm{Tr}[\sum_l F_l^\dagger F_l] = d$, $d = \dim(\mathsf{H})$). Moreover, the state Φ is symmetric iff

$(\mathscr{F}^{-1})^t = \mathscr{F}^{-1}$, corresponding to the operator identity $E\Phi E = \Phi$, E denoting the swap operator. According to Eq. (10) the transposed with respect to Φ is given by

$$\tau_\Phi(\mathscr{A}) = \mathscr{F}\mathscr{A}^t\mathscr{F}^{-1} \tag{43}$$

The canonical basis of eigenvectors $\{C_j\}$ of the bilinear form Φ must satisfy the identity

$$s_j\delta_{ij} = \Phi(C_i, C_j) = \mathrm{Tr}[(C_i \otimes C_j)\sum_l |F_l\rangle\!\rangle\langle\!\langle F_l|] = \mathrm{Tr}[C_j{}^t\mathscr{F}^\dagger(C_i)]. \tag{44}$$

Upon multiplying by C_j and summing over the index j in Eq. (44) we obtain [11]

$$s_iC_i = \sum_j \mathrm{Tr}[C_j{}^t\mathscr{F}^\dagger(C_i)]C_j =: \mathscr{C}\mathscr{F}^\dagger(C_i) \tag{45}$$

where $\mathscr{C} := \sum_j \mathrm{Tr}[C_j{}^t\cdot]C_j$. Identity (44) is equally satisfied by the set $\{C_i^\dagger\}$ with the same eigenvalue. Therefore, it is always possible to choose the operators C_j as selfadjoint, and $\mathscr{C}^\dagger \equiv \mathscr{C}$. It is also easy to check that $\mathscr{C}^t = \mathscr{C}$, since

$$\begin{aligned}\mathscr{C}\otimes\mathscr{I}(|I\rangle\!\rangle\langle\!\langle I|) &= \sum_j C_j \otimes \mathrm{Tr}_1[C_j{}^t \otimes I|I\rangle\!\rangle\langle\!\langle I|] = \sum_j C_j \otimes C_j \\ &= \sum_j \mathrm{Tr}_2[I \otimes C_j{}^t|I\rangle\!\rangle\langle\!\langle I|] \otimes C_j = \mathscr{I}\otimes\mathscr{C}(|I\rangle\!\rangle\langle\!\langle I|).\end{aligned} \tag{46}$$

Using completeness of $\{C_j\}$ and their self-adjointness, it is easy to see that

$$\mathscr{C}(X) = \sum_j \mathrm{Tr}[C_j{}^tX]C_j = \sum_j \mathrm{Tr}[C_jX^t]C_j = \sum_j \mathrm{Tr}[C_j^\dagger X^t]C_j = X^t, \tag{47}$$

namely

$$\mathscr{C} = (\cdot)^t, \tag{48}$$

and using Eq. (46) one can see that $\sum_j C_j \otimes C_j = E$ and $\{C_j\}$ are Hilbert-Schmidt orthonormal. Clearly $\mathscr{C}^2 = \mathscr{I}$, $\mathscr{C}\mathscr{M}\mathscr{C} = \mathscr{M}^*$, i. e. $\mathscr{C} = \mathscr{Z}_\Omega$. According to (23) this will then guarantee that the adjoint will be independent on the faithful state Φ. The map $\mathscr{C}\mathscr{F}^\dagger$ acting on C_i gives their complex conjugated, and since $\{C_i\}$ is a selfadjont basis of the real linear space of selfadjoint operators, $\mathscr{C}\mathscr{F}^\dagger$ is the complex conjugation over all selfadjoint operators, namely [12]

$$\mathscr{Z}_\Phi = \mathscr{C}\mathscr{F}^\dagger = \mathscr{F}\mathscr{C}. \tag{49}$$

The complex conjugation obeys the symmetry $\tau_\Phi(\mathscr{Z}_\Phi) = \mathscr{Z}_\Phi$ which is needed for a proper definition of the adjoint. Indeed

$$\tau_\Phi(\mathscr{Z}_\Phi) = \mathscr{F}\mathscr{Z}_\Phi{}^t\mathscr{F}^{-1} = \mathscr{F}\mathscr{C}\mathscr{F}\mathscr{F}^{-1} = \mathscr{F}\mathscr{C} = \mathscr{Z}_\Phi. \tag{50}$$

Since, by definition, the map $\mathscr{Z}$ is involutive, one has

$$\mathscr{I} = \mathscr{F}\mathscr{C}\mathscr{F}\mathscr{C} = \mathscr{F}\mathscr{F}^* = \mathscr{F}\mathscr{F}^\dagger \tag{51}$$

whence

$$\mathscr{F}^{-1} = \mathscr{F}^* = \mathscr{F}^\dagger. \tag{52}$$

Finally, the adjoint of a map $\mathscr{A}$ is just the usual adjoint, since

$$\varsigma_\Phi\tau_\Phi(\mathscr{A}) = \mathscr{Z}_\Phi\mathscr{F}\mathscr{A}^t\mathscr{F}^{-1}\mathscr{Z}_\Phi = \mathscr{C}\mathscr{A}^t\mathscr{C} = \mathscr{A}^{t*} = \mathscr{A}^\dagger, \tag{53}$$

or, equivalently,

$$\tau_\Phi\varsigma_\Phi(\mathscr{A}) = \mathscr{F}(\mathscr{Z}_\Phi\mathscr{A}\mathscr{Z}_\Phi)^t\mathscr{F}^{-1} = \mathscr{F}\mathscr{C}\mathscr{F}\mathscr{A}^t\mathscr{C}\mathscr{F}\mathscr{F}^{-1} = \mathscr{C}\mathscr{A}^t\mathscr{C} = \mathscr{A}^\dagger. \tag{54}$$

In Table 1 I summarize the most relevant identities and definitions.

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[11] In the present quantum context the notation $\mathscr{F}^\dagger(X)$ corresponds to the Heisenberg picture $\underline{\mathscr{X}} \circ \mathscr{F}$, with X selfadjoint operator representing the generalized effect $\mathscr{X}$.

[12] On the other hand, for a generic self-adjoint operator it is easy to check that

$$\mathscr{Z}_\Phi(A) = \sum_k \Phi(C_k, A)C_k = \sum_{kl} \mathrm{Tr}[F_l^*AF_l{}^tC_k{}^t]C_k = \sum_{kl}\mathrm{Tr}[F_l^\dagger AF_lC_k{}^t]C_k = \mathscr{C}\mathscr{F}^\dagger(A).$$

TABLE 1. Summary of most relevant identities and definitions

object	definition	identities
Φ	$\sum_l \vert F_l\rangle\!\rangle\langle\!\langle F_l\vert$	$E\Phi E = \Phi$
$\mathscr{F}$	$\sum_l F_l \cdot F_l^\dagger$	$\mathscr{F} = \mathscr{F}^t, \mathscr{F}^{-1} = \mathscr{F}^* = \mathscr{F}^\dagger$
$\mathscr{C}$	$\sum_j \mathrm{Tr}[C_j{}^t \cdot] C_j$	$\mathscr{C} = (\cdot)^t = \mathscr{C}^\dagger = \mathscr{C}^t, \mathscr{C}\mathscr{M}\mathscr{C} = \mathscr{M}^*$
$\mathscr{Z}_\Phi$	$\sum_j \Phi(C_j, \cdot) C_j$	$\mathscr{Z}_\Phi = \mathscr{C}\mathscr{F}^\dagger = \mathscr{F}\mathscr{C}$
$\tau_\Phi(\cdot)$		$\tau_\Phi(\mathscr{M}) = \mathscr{F}\mathscr{M}^t\mathscr{F}^{-1}$
$\varsigma_\Phi(\cdot)$		$\varsigma_\Phi(\mathscr{M}) = \mathscr{Z}_\Phi \mathscr{A} \mathscr{Z}_\Phi$, $\tau_\Phi(\mathscr{Z}_\Phi) = \mathscr{Z}_\Phi$
$\mathrm{ad}_\Phi(\cdot)$		$\tau_\Phi(\varsigma_\Phi(\mathscr{A})) = \varsigma_\Phi(\tau_\Phi(\mathscr{A})) = \mathscr{A}^\dagger$
${}_\Phi\langle\mathscr{A}\vert\mathscr{B}\rangle_\Phi$	$\Phi(\mathscr{A}^\dagger, \tau_\Phi(\mathscr{B}))$	${}_\Phi\langle\mathscr{A}\vert\mathscr{B}\rangle_\Phi = \frac{1}{d}\sum_l \langle\!\langle F_l\vert\check{A}^\dagger\check{B}\vert F_l\rangle\!\rangle$

Explicitely, the GNS representation is given by

$$ {}_\Phi\langle\mathscr{A}|\mathscr{B}\rangle_\Phi := \Phi(\mathscr{A}^\dagger, \tau_\Phi(\mathscr{B})) = \frac{1}{d}\sum_l \langle\!\langle F_l|\check{A}^\dagger\check{B}|F_l\rangle\!\rangle, \tag{55}$$

where for any CP map $\mathscr{A} = \sum_i A_i \cdot A_i^\dagger$ one has $\check{A} := \sum_i A_i \otimes A_i^*$ (we remind the normalization $\mathrm{Tr}[\sum_l F_l^\dagger F_l] = d$ of state Φ in terms of the Kraus operators of $\mathscr{F}$). For trace-preserving $\mathscr{F}$ one would obtain the same scalar product as in Eq. (38), i. e. ${}_\Phi\langle\mathscr{A}|\mathscr{B}\rangle_\Phi := \mathrm{Tr}[P_{\mathscr{A}^\dagger} P_{\mathscr{B}^\dagger}]$, however, since $\mathscr{F}^{-1} = \mathscr{F}^*$ is also trace preserving, the only possibility would be $\mathscr{F} = U \cdot U^\dagger$ unitary, and with the additional constraint $U = U^t$ coming from symmetry of Φ.

6.3. The most general quantum scalar product

We start now from the most general scalar product between two quantum transformations and show that it must be of the form (55). The most general form of scalar product between two operators A and B in $\mathsf{B}(\mathsf{H})$ is

$$\varphi(A^\dagger B) = \sum_j \langle \upsilon_j|A^\dagger B|\upsilon_j\rangle, \quad \sum_j \langle\upsilon_j|\upsilon_j\rangle = 1 \tag{56}$$

where normalization corresponds to $\varphi(I) = 1$. For quantum transformations the most general scalar product can be constructed upon regarding transformations as operators on $\mathsf{B}(\mathsf{H})$ (in infinite dimensions, more precisely, as operators on the Hilbert space of the Hilbert-Schmidt operators). Therefore, upon considering a complete set of operators $\{E_i\}$, one has

$$(\mathscr{B}, \mathscr{A}) = \sum_i \langle\!\langle \mathscr{B}(E_i)|\mathscr{A}(E_i)\rangle\!\rangle = \sum_i \langle\!\langle E_i|\check{B}^\dagger\check{A}|E_i\rangle\!\rangle, \quad \mathrm{Tr}\left(\sum_i E_i^\dagger E_i\right) = 1, \tag{57}$$

which is exactly of the general form (55). Notice that the general form (55) corresponds to a state Φ that is mixed, being the convex combination $\Phi = \sum_i \mathrm{Tr}[F_i^\dagger F_i] |\tilde{F}_i\rangle\!\rangle\langle\!\langle\tilde{F}_i|$, with $\tilde{F}_i := F_i/\sqrt{\mathrm{Tr}[F_i^\dagger F_i]}$.

7. CONCLUSIONS

In conclusion I want to emphasize that the fact that Postulates 1 and 2 imply a C^*-algebra representation for transformations, and with the correct Born-rule pairing and the correct dimensionality for the tensor-product structure of bipartite systems, is not sufficient to assert that the only possible theory derived from the postulates is Quantum Mechanics. Indeed, as for the general C^*-algebras of operators on Hilbert spaces, Classical Mechanics is also included as special case, corresponding to Abelian A, and, more generally, a combination of both Quantum and Classical in a direct sum of irreducible algebra representations, such as in the presence of constant of motions and/or super-selection rules. Indeed preliminary analysis [7] show that more general theories can satisfy both postulates, such as the non-local no-signaling probabilistic theories generally referred to as *PR boxes* [8]. This is the case, for example, of the model in Ref. [9], which possesses a symmetric faithful state, however with $\dim(\mathfrak{P}_{\mathbb{R}}) = 3$, which cannot be quantum.

As regards additional postulates selecting Quantum Mechanics from the set of theories admitting C^*-algebras representations, one may adopt Postulate 4 in Ref. [1] concerning the possibility of achieving an informationally complete observable by means of a perfectly discriminating observable over system+ancilla. However, such postulate may look quite *ad hoc*, being essentially a restatement of existence of Bell measurements (Bell measurements are locally informationally complete for one system for almost every state-preparation of the other system). Alternative candidates for the quantum-extracting postulate are under study, considering what is specific of the quantum C^*-algebra, e. g. the fact that in the quantum case the C^*-algebra of transformations $\mathcal{A}$ is a kind of *multiplier algebra* [10] of the C^*-algebra $\mathsf{B}(\mathsf{H})$.

I want to stress that the dimensionality identity in Sect. 5 concerning only identical independent systems could be generalized to the case of different systems. This, however, will need to consider transformations between different systems. Thus, also the symmetry of the faithful state must be relaxed, upon considering a suitable transformation that maps the largest to the smallest system. Finally, the faithfulness condition itself may be relaxed, obtaining a generally unfaithful C^*-algebra representation. Thus the C^*-algebra representation of transformations will be just equivalent to the probabilistic framework endowed with the postulated existence of dynamically independent systems. A complete analysis of this direction will be the subject of a forthcoming publication [11].

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