Quantum Circuit Architecture

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We present a method for optimizing quantum circuits architecture, based on the notion of a quantum comb, which describes a circuit board where one can insert variable subcircuits. Unexplored quantum processing tasks, such as cloning and storing or retrieving of gates, can be optimized, along with setups for tomography and discrimination or estimation of quantum circuits.

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Quantum mechanics plays a crucial role in the technology of high precision and high sensitivity, e.g., in frequency standards [1], quantum lithography [2], two-photon microscopy [3], clock synchronization [4], and reference-frame transfer [5]. In these applications, the problem is to achieve high precision in (i) determining parameters and (ii) executing transformations that depend on unknown parameters. Since the parameters are generally encoded by a transformation, as in the whole class of quantum metrology problems [6], and since the estimation itself can be considered as a special case of transformation (with classical output), both tasks (i) and (ii) can be reduced to the general problem of executing a desired transformation depending on an unknown transformation. Taking into account the possibility of exploiting N uses of the unknown transformation, the problem is to build a quantum circuit that has N circuits as input, and achieves the desired transformation as an output. This is what we call a quantum circuit board.

A quantum circuit board is a network of gates in which there are N slots with open ports for the insertion of N variable subcircuits (see Fig. 1). Since generally it is impossible even in principle to achieve the desired transformation exactly, the main task here is to optimize the circuit board according to a given figure of merit. A typical example is the optimal cloning of an undisclosed transformation U, which will be operated by a board with N slotted uses of U, and achieving overall in-out transformation which is the closest possible to $U^{\otimes M}$ with $M > N$. We emphasize that generally the overall in-out transformation of the board and of the slotted circuits can be of any kind, including measurements and state-preparations, and the slotted transformations can be different from each other.

In previous literature, the only case of circuit-board optimization that has been considered is that of phase estimation [7]. In other applications, such as discrimination and estimation of unitary transformations with N uses, optimization has been carried out only for fixed architectures—i.e., with uses either in parallel [8,9], or in sequence [10]—since no systematic optimization method for variable architecture was available. On the other hand, the problem of deriving the optimal circuit board for channel tomography is still beyond the current possibilities of available optimization approaches.

In this Letter, we present a complete method for optimizing the architecture of quantum circuit boards. After providing a convenient description of circuit connectivity, we introduce the notion of quantum comb, which describes all possible transformations operated by a quantum circuit board, and generalizes the notion of quantum channel to the case where the inputs are quantum circuits, rather than quantum states. We then present the optimization method, based on the convex structure of the set of quantum combs. The method allows one to reduce the apparently untreatable problem of optimal circuit architecture to the optimization of a single positive operator with linear constraints. Since the positive operator summarizes all the relevant features of the circuit, our method automatically determines the optimal causal disposition of the variable slots. We will give several applications in which the present approach dramatically simplifies the solution of the problem.

A quantum circuit operates a transformation from input to output, and is graphically represented by a box with input and output wires symbolizing the respective quantum systems. Systems corresponding to different wires are generally different, and may also vary from input to output. Let us associate Hilbert spaces $\mathsf{H}_{\rm in}$ ($\mathsf{H}_{\rm out}$) to all input (output) wires, and denote by $\rho_{\rm in}$ ($\rho_{\rm out}$) the corresponding states. The action of the circuit is generally probabilistic; i.e., different in-out transformations can randomly occur, as in a measurement process. Each transformation is described by a linear map $\rho_{\rm in} \to \mathcal{C}(\rho_{\rm in}) = k\rho_{\rm out}$, with the proportionality factor $0 \le k = \mathrm{Tr}[\mathcal{C}(\rho_{\rm in})] \le 1$ giving the probability that $\mathcal{C}$ occurs on state $\rho_{\rm in}$. To describe a legitimate quantum transformation, the map $\mathcal{C}:\mathsf{Lin}(\mathsf{H}_{\rm in}) \to \mathsf{Lin}(\mathsf{H}_{\rm out})$ [11] has to be completely positive (CP) and trace

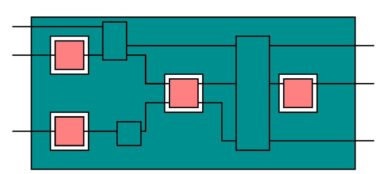

FIG. 1 (color online). A quantum circuit board.

nonincreasing. Trace-preserving maps—i.e., deterministic transformations—are called quantum channels. Notice that a map $\mathcal{C}$, rather than representing a specific circuit, is univocally associated to the equivalence class of all circuits performing the same in-out transformation.

The linear map $\mathcal{C}$ can be conveniently rewritten using the so-called "Choi-Jamiołkowski" representation [12], corresponding to the following one-to-one correspondence between linear maps $\mathcal{C}:\mathsf{Lin}(\mathsf{H}_{\mathrm{in}}) \to \mathsf{Lin}(\mathsf{H}_{\mathrm{out}})$ and linear operators $C \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}})$ given by

$$C = \mathrm{Choi}(\mathcal{C}) := \mathcal{C} \otimes \mathcal{I}(|\Omega\rangle\langle\Omega|), \tag{1}$$

$$\mathcal{C}(\rho) = \mathrm{Choi}^{-1}(C)(\rho) := \mathrm{Tr}_{\mathrm{in}}[(I_{\mathrm{out}} \otimes \rho^T)C], \tag{2}$$

where $\mathcal{I}$ is the identity map, $|\Omega\rangle$ is the un-normalized maximally entangled state $|\Omega\rangle = \sum_n |n\rangle|n\rangle \in \mathsf{H}_{\mathrm{in}}^{\otimes 2}$, and T denotes transposition with respect to the orthonormal basis $\{|n\rangle\}$ for H_{in}. The map $\mathcal{C}$ is CP if and only if the operator C—called Choi operator—is positive [13].

Two quantum circuits can be connected in all the ways allowed by the physical matchings between input and output wires (see, e.g., Fig. 2, where the wires labeled $\boldsymbol{d}$ are connected): a connection will result in the composition of the corresponding CP maps, and hence of the corresponding Choi operators. Since building a quantum network means connecting many circuits, it is crucial to have a handy way to describe circuit connectivity with minimum overhead of notation. We provide here three simple rules that accomplish this goal:

Rule 1 (Labelling) Each quantum wire is marked with a different label, except for wires that are connected, which are identified with the same label.

Rule 2 (Multiplication) The multiplication of two Choi operators $A \in \mathsf{Lin}(\mathsf{H}_{a,b,c,\boldsymbol{d}})$ and $B \in \mathsf{Lin}(\mathsf{H}_{\boldsymbol{d},e,f,g})$ is regarded in the tensor fashion, i.e., $AB = (A \otimes I_{e,f,g})(I_{a,b,c} \otimes B)$.

Rule 3 (Composition) The connection of two circuits with Choi operators A and B—acting on Hilbert spaces labeled according to Rule 1—yields a new circuit with Choi operator C given by the link product

$$C = A * B = \mathrm{Tr}_{\mathsf{J}}[A^{\theta_{\mathsf{J}}} B], \tag{3}$$

θ_{J} denoting partial transposition over the Hilbert space J of the connected wires, and the multiplication in square brackets following Rule 2.

Rule 3 follows from Eqs. (1) and (2). Notice that due to invariance of trace under cyclic permutations, the link product is commutative: $A * B = B * A$. Using it, the action of a linear map $\mathcal{C}$ on a state ρ in Eq. (2) can be rewritten as $\mathcal{C}(\rho) = C * \rho$. Assembling many circuits $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ yields a quantum network whose Choi operator is simply given by $C = C_1 * C_2 * \ldots * C_k$.

We are now ready to treat quantum circuit boards. To start with, we consider the case of a deterministic circuit board, i.e., a network of quantum channels with N open slots for the insertion of variable subcircuits. It is clear that by reshuffling and stretching the internal wires, any circuit board can be reshaped in the form of a "comb," with an ordered sequence of slots, each between two successive teeth, as in Fig. 3. The order of the slots is the causal order induced by the flow of quantum information in the circuit board. We label the input systems (entering the board) with even numbers $2n$, and the corresponding output systems (exiting the board) with odd numbers $2n + 1$, with n ranging from 0 to N.

A quantum comb with N slots is clearly equivalent to a concatenation of $N + 1$ channels with memory, which is in turn equivalent to a causal network, namely, a network where the quantum state of the output systems up to time n does not depend on the state of the input systems at later times $n' > n$, with $n, n' \in \{0, 1, \ldots, N\}$ [14]. The causal network can be easily obtained by redrawing the comb as an equivalent circuit with all inputs on the left and all outputs on the right, as in Fig. 4. We define the Choi operator of a quantum comb as the Choi operator R of the corresponding causal network. In terms of the Choi operator R, causality is equivalent to a set of linear constraints

$$\begin{aligned} \mathrm{Tr}_{2n+1}[R^{(n)}] &= I_{2n} \otimes R^{(n-1)}, \qquad n = 0, \ldots N, \\ R^{(N)} &\equiv R, \qquad R^{(-1)} = 1, \end{aligned} \tag{4}$$

where Tr_{2n+1} denotes the partial trace over the Hilbert

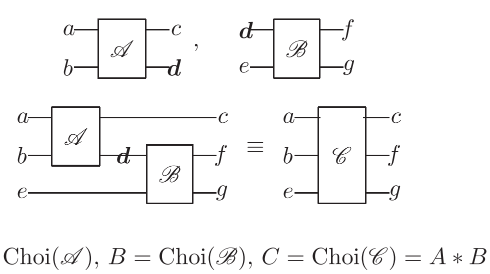

FIG. 2. Connection of two quantum circuits $\mathcal{A}$ and $\mathcal{B}$. Wires are labeled according to Rule 1. The Choi operator of the resulting circuit $\mathcal{C}$ is given by the link product of Rule 3.

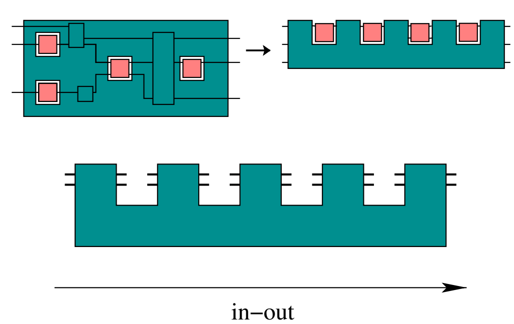

FIG. 3 (color online). Every circuit-board can be reshaped in form of a "comb," with an ordered sequence of slots, each between two successive teeth. The pins represent quantum systems, entering or exiting from the board (the horizontal arrow represents the quantum information flow).

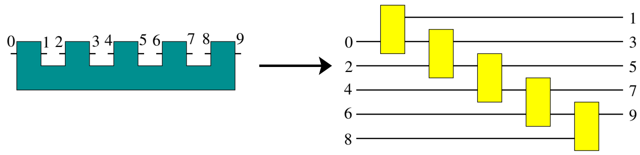

FIG. 4 (color online). Each quantum comb is equivalent to a causal network, with all inputs on the left and all outputs on the right. The Choi operator of a comb is the Choi operator of the corresponding causal network.

space H_{2n+1} of the output wire labeled $2n+1$, I_{2n} the identity operator over the Hilbert space H_{2n} of the input wire labeled $2n$, $R^{(n)} = \mathrm{Choi}(\mathcal{C}^{(n)})$, and $\mathcal{C}^{(n)}$ is the map of the $(n+1)$-subnetwork from the first $n+1$ inputs to the first $n+1$ outputs. Precisely, we have the following:

Theorem 1.—Every positive operator $0 \leq R \in \mathrm{Lin}(\otimes_{j=0}^{2N+1}\mathsf{H}_j)$ satisfying the linear constraints (4), is the Choi operator of a deterministic quantum comb.

Proof.—By definition, it is enough to show that any operator $R \geq 0$ normalized as in Eq. (4) is the Choi operator of a causal network. A causal network with $N+1$ input/output pairs is described by a family of channels $\mathcal{C}^{(n)}$, $n = 0, 1, \ldots, N$ with the property $\mathrm{Tr}_{2n+1}[\mathcal{C}^{(n)}(\rho^{(n)})] = \mathcal{C}^{(n-1)}(\mathrm{Tr}_{2n}[\rho^{(n)}])$ for any state $\rho^{(n)}$ of the first $n+1$ input systems. Using the correspondence of Eq. (2), one can easily see that this is equivalent to the normalization of Eq. (4).

A quantum comb transforms a series of N input circuits $\mathcal{C}_1, \ldots, \mathcal{C}_N$ into an output circuit $\mathcal{C}'$ depending on them [Fig. 5(a)]. This transformation of circuits corresponds to an N-linear CP-map that sends the input Choi operators into the ouput Choi operator according to $C' = C_1 * \ldots * C_N * R$, with R the Choi operator of the comb. We call the mapping between circuits $\{\mathcal{C}_1, \ldots, \mathcal{C}_N\} \mapsto C'$ supermap as it sends channels into channels, rather than states into states. Notice that, depending on the number of slots that are saturated, a quantum comb can transform a series of circuits into a comb [Fig. 5(b)], or, more generally, a comb into a comb [Fig. 5(c)]. A quantum comb can realize many possible mappings, all obtained by different link products with its Choi operator R. Therefore, the quantum comb can be completely identified with its Choi operator. Remarkably, also the converse is true: any abstract supermap sending channels into channels in a CP fashion can be physically realized by a quantum comb [15].

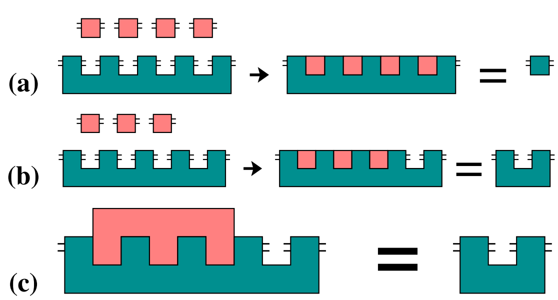

FIG. 5 (color online). A quantum comb realizes different transformations of quantum circuits, namely, it can send (a) a series of channels into a channel, (b) a series of channels into a comb, or (c) an input comb into an output comb.

The tools presented above provide a powerful method for optimizing quantum circuit architecture. Suppose we want to design a circuit-board maximizing some convex figure of merit, e.g., the fidelity of the output circuit $\mathcal{C}'$ with a desired unitary gate $\mathcal{U}$. In our framework, the optimization of the board architecture is reduced to the search of the optimal operator $R \geq 0$ with the linear constraints (4). This is a standard problem of convex optimization, for which efficient algorithms are known. Basically, we need to implement the search on the extremal points of the convex set of Choi operators. Moreover, the complexity of the search can be dramatically reduced by exploiting additional constraints, e.g., symmetry properties of the circuit board. The optimal Choi operator will finally single out the optimal architecture, automatically deciding if the N slots of the circuit-board have to be connected in a causal order or in parallel, or in any combination of the two.

We illustrate our method in some concrete applications. The first is the optimal universal cloning of unitary transformations, i.e., the problem of designing a quantum board that optimally achieves the $N \to M$ cloning of an unknown unitary $U \in \mathbb{SU}(d)$ in dimension d. The board has N slots containing N identical uses of the unknown unitary U and performs a transformation which is the closest possible to $U^{\otimes M}$. Evaluation for $1 \to 2$ cloning [16] using as figure of merit the channel fidelity averaged over all possible unitaries leads to optimal value $F = (d + \sqrt{d^2 - 1})/d^3$, significantly higher than the classical threshold reached by the optimal estimation of a unitary $F^{\mathrm{est}} = 6/d^4$ for $d > 2$, $F^{\mathrm{est}} = 5/16$ for qubits, thus showing the advantage of coherent quantum information processing over any classical cloning strategy.

A second interesting application is the storage and retrieval of an undisclosed unitary transformation U from N uses, also called optimal quantum-algorithm learning. The problem arises from the need of running an undisclosed algorithm (available for N uses) on an input state ψ which will be available at later time. To this purpose, one can slot the N uses of U in a quantum circuit board, put the output state of the board in a quantum memory, and, when the input state will be available, use the memory to recover the unitary. The series storing or retrieving is represented by the quantum comb in Fig. 6, which can be cut into two

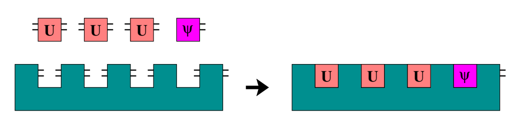

FIG. 6 (color online). Quantum-algorithm learning. One wants to run an undisclosed unitary U on a quantum state ψ which is available after the lapse of time in which the uses of U are available.

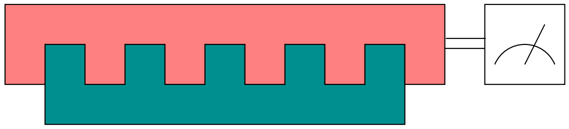

FIG. 7 (color online). Quantum tester: comb for measuring quantum combs, e.g., memory channels and sequences of unitaries.

parts: a storing comb including only the uses of U, and a retrieving comb including ψ (the output state of the first part is stored in a quantum memory and is then fed in the second part).

An evaluation for $N = 1, 2$ gives optimal average channel fidelities $F = \frac{2}{d^2}$ and $F = \frac{3}{d^2}$, respectively, coinciding with the value attained by the optimal estimation of unitaries. In these two cases, the optimal storing algorithm is classical.

The present method can be easily extended to the optimization of probabilistic circuit boards, containing measuring devices that produce different transformations depending on random outcomes. The probabilistic comb corresponding to outcome i will have Choi operator R_i, with the sum over all outcomes $\sum_i R_i = R$ giving the Choi operator of a deterministic comb [17]. A special case is that of a comb that tests another comb—the so-called quantum tester (see Fig. 7)—e.g., a device that tests sequences of channels, or more generally memory channels. The probability distribution of the outcomes can then be written in the form of a generalized Born rule $p_j = \mathrm{Tr}[RP_j]$ where R is the Choi operator of the tested comb, and $\{P_i\}$ is the mathematical representation of the tester, with $P_i \geq 0$ for all outcomes i, but $\sum_i P_i = I_{2N-1} \otimes \Xi^{(N)}$, with $\mathrm{Tr}_{2n}[\Xi^{(n)}] = I_{2n-1} \otimes \Xi^{(n-1)} \quad \forall 1 < n \leq N$ and $\mathrm{Tr}[\Xi^{(0)}] = 1$. The tester is thus a generalization of the concept itself of POVM, with the Choi playing the role of the state in the Born rule. In Ref. [18], it is shown that the optimal discrimination of two memory channels requires a tester as in Fig. 7, and cannot be achieved by the customary optimization of the input state and the final measurement. Moreover, the theory of testers allows one to determine the optimal tomographic setup for quantum channels, which is simply realizable in the lab, and was unknown before [19].

In addition to the above applications, quantum combs are also the precise description of single-party strategies in quantum protocols, i.e., games, cryptography, and algorithms. In the last case, the slots represent calls to the oracles, and the present method provides an ideal framework to address minimization of the number of calls, with the possibility of achieving provably optimal algorithms, e.g., in a generalized Simon algorithm, which resorts to discrimination of classes of quantum oracles [20,21].

In conclusion, we introduced a new method for optimization of quantum networks and illustrated its effectiveness in several applications in which our theory solves problems that could not even be addressed otherwise. As main examples, comb theory allows us to optimize cloning, storage retrieving, discrimination, estimation, and tomographic characterization of quantum circuits.

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