No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels

Giacomo Mauro D'Ariano, Stefano Facchini, and Paolo Perinotti

QUIT Group, Dipartimento di Fisica "A. Volta," INFN Sezione di Pavia, via Bassi 6, 27100 Pavia, Italy
(Received 20 July 2010; published 4 January 2011)

A bipartite quantum channel represents the interaction between systems, generally allowing for the exchange of information. A special class of bipartite channels is the no-signaling ones, which do not allow for communication. Piani *et al.* [Phys. Rev. A **74**, 012305 (2006)] conjectured that all no-signaling channels are mixtures of entanglement breaking and localizable channels, which require only local operations and entanglement. Here we provide the general realization scheme, and give a counterexample to the conjecture, achieving no-signaling superquantum correlations while preserving entanglement.

Causality is the basic assumption of science, the building block of any mechanism, and any prediction scheme [1]. It is the gray eminence of physical theories, taking apparently different forms, such as retarded potentials in classical physics, Minkowskian causality in relativity, and (anti)commutation relations in quantum field theory. The modern paradigm of causality is communication, where we identify the causal relation with information exchange. Causality should not be confused with determinism: indeed, any communication scheme from Alice to Bob corresponds to a dependence of the outcome probability distributions at Bob's location on Alice's choice. It is easy to recognize that such schemes encompass all customary definitions of causality, including determinism as a very special case. In synthesis, we define causality as the dependence of a probability distribution on a choice.

In the past, quantum entanglement has been claimed as a resource for communication [2], regarding Alice's choice of local measurement as a way of changing Bob's probabilities—the spooky action at a distance of Einstein [3]. The impossibility of communicating by local operations—today commonly referred to as no signaling—is instead an immediate consequence of causality of the theory, as proved in Ref. [4].

In order to have a causal relation between two systems A (Alice) and B (Bob) one needs an interaction between them. In quantum theory such interaction is represented by a bipartite channel for A and B, with communication from A to B corresponding to the dependence of the local output state of system B on the choice of the input state of system A. Indeed, one can generalize the scheme to the case of A and B at the input being different from A' and B' at the output, considering the causal relation, e.g., from A to B'. More generally we can include the case of one-dimensional systems, thus recovering also the situation of monopartite channels (the case of both inputs and/or both outputs one dimensional is uninteresting, since there is no input and/or no output then). While monopartite channels have trivial causality properties—the only no-signaling monopartite channels being those that prepare a fixed state

irrespectively of the input state—bipartite channels provide the minimal nontrivial interaction scenario. For simplicity we will restrict to finite dimensions, and use the same capital Roman letter to denote the system and the corresponding Hilbert space, writing $\mathcal{L}(A)$ for the space of operators on A. The graphical representation of the bipartite quantum channel $\mathcal{C}:\mathcal{L}(A \otimes B) \rightarrow \mathcal{L}(A' \otimes B')$ is the following:
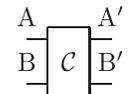
$$
\begin{array}{c}
\text{A} \quad\quad \text{A}' \\
\boxed{\mathcal{C}} \\
\text{B} \quad\quad \text{B}'
\end{array}
\tag{1}
$$

The natural question is now which interactions do not allow for communication between input and output. For example, one cannot achieve signaling by local operations using entanglement—such bipartite channels are called localizable. However, as shown in Ref. [5], not every no-signaling channel is localizable (see also Definition 1), and the problem is how to generate "superquantum" correlations—i.e., stronger than those arising from entanglement—without signaling, as for PR boxes [6]. In the same reference it has been conjectured that all semicausal channels (namely, no signaling from B to A', but not necessarily from A to B') are also semilocalizable, namely, they are of the form
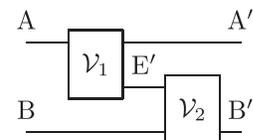
$$
\begin{array}{c}
\text{A} \quad\quad\quad\quad\quad\quad \text{A}' \\
\boxed{\mathcal{V}_1} \; \text{E}' \\
\quad\quad \boxed{\mathcal{V}_2} \; \text{B}' \\
\text{B}
\end{array}
\tag{2}
$$

for some system E' and suitable quantum channels $\mathcal{V}_1$ and $\mathcal{V}_2$. Such conjecture has later been proved in Ref. [7]. An alternative proof was given in Ref. [8], where the authors also proposed the following:

Conjecture 1.—All no-signaling channels are mixtures of entanglement-breaking and localizable channels.

The conjecture was based on the only known quantum realization of a PR box, which was made with an entanglement-breaking channel, i.e., a channel which destroys the entanglement between its input system and any other system. Such conjecture, however, implicitly forbids truly coherent superquantum correlations. This corresponds to perfect monogamy of correlations, in the sense that when the channel violates the Cirel'son bound [9] the entanglement of the input systems with other ones is broken. We will show that Conjecture 1 is false, allowing for more flexibility, with a trade-off between generated correlations and preserved entanglement, and with a violation of the Cirel'son bound achieved coherently, in the full range between the quantum bound and the maximum possible correlation. We will also provide the general realization scheme for the no-signaling bipartite channel, along with a concrete counterexample to Conjecture 1.

We will stick on the graphical representation of a bipartite quantum channel in Eq. (1). By "quantum channel" we mean a completely positive, trace-preserving map between the density-matrix space of the input systems and that of the output systems. The preparation of a state ρ and measurement of a POVM $\{P_x\}$ on system A are special classes of channels, with one-dimensional input and output space, respectively, graphically represented as

$$\boxed{\rho} \!-\!\! \overset{A}{} \qquad \overset{A}{} \!-\!\! \boxed{P_x} . \qquad (3)$$

We will use the bijection between states and operators $A = \sum_{mn} A_{mn}|m\rangle\langle n| \leftrightarrow |A\rangle\rangle = \sum_{mn} A_{mn}|m\rangle|n\rangle$ summarized by the identity $|A\rangle\rangle = (A \otimes I)|I\rangle\rangle$, where $|I\rangle\rangle = \sum_n |n\rangle|n\rangle$ is the (unnormalized) maximally entangled state. It will also be useful to introduce the Choi-Jamiołkowski isomorphism between channels $\mathcal{C}\!:\mathcal{L}(A) \to \mathcal{L}(B)$ and positive operators on $B \otimes A$: $R_{\mathcal{C}} := \mathcal{C} \otimes I_A(|I\rangle\rangle\langle\langle I|)$, $\mathcal{C}(\rho) = \mathrm{Tr}_A[(I \otimes \rho^T)R_{\mathcal{C}}]$, where ρ^T denotes the transposition of the operator ρ with respect to the orthonormal basis $|n\rangle$.

We are now in position to make the above-mentioned concepts more precise:

Definition 1.—The channel $\mathcal{C}\!:\mathcal{L}(A) \otimes \mathcal{L}(B) \to \mathcal{L}(A') \otimes \mathcal{L}(B')$ is "localizable" if it can be realized by local operations on $A \otimes E_A$ and $B \otimes E_B$ with a shared (possibly entangled) ancilla $E_A \otimes E_B$ in a state ρ without communication:

$$\underset{B}{\overset{A}{\boxed{\mathcal{C}}}} \!\!\begin{array}{c} A' \\ B' \end{array} = \begin{array}{c} A \qquad\qquad A' \\ \boxed{\rho}\begin{array}{c} E_A \\ E_B \end{array} \begin{array}{c} \boxed{\mathcal{G}_A} \\ \\ \boxed{\mathcal{G}_B} \end{array} \begin{array}{c} \\ B' \end{array} \\ B \end{array}. \qquad (4)$$

Definition 2.—A bipartite quantum channel $\mathcal{C}\!:\mathcal{L}(A) \otimes \mathcal{L}(B) \to \mathcal{L}(A') \otimes \mathcal{L}(B')$ is "$A \nrightarrow B'$ no signaling" if there exists $S_{BB'}$ such that $\mathrm{Tr}_{A'}[R_{\mathcal{C}}] = I_A \otimes S_{BB'}$. We say that $\mathcal{C}$ is

"no signaling" if it is both $A \nrightarrow B'$ no signaling and $B \nrightarrow A'$ no signaling.

The following theorem holds:

Theorem 1.—The following are equivalent: 1. The channel $\mathcal{C}\!:\mathcal{L}(A \otimes B) \to \mathcal{L}(A' \otimes B')$ is no signaling. 2. There are two equivalent d-dimensional quantum systems E_A, E_B, a bipartite state ρ of $E_A \otimes E_B$, instruments $\{\mathcal{C}_A^{(x)}\}_{x \in X}$ and $\{\mathcal{D}_B^{(x)}\}_{x \in X}$ with outcome space X, and channels $\mathcal{C}_B^{(x)}$, $\mathcal{D}_A^{(x)}$ for each $x \in X$ with

$$\mathcal{C}_A^{(x)}\!:\mathcal{L}(A \otimes E_A) \to \mathcal{L}(A'), \qquad \mathcal{C}_B^{(x)}\!:\mathcal{L}(B \otimes E_B) \to \mathcal{L}(B')$$

$$\mathcal{D}_B^{(x)}\!:\mathcal{L}(B \otimes E_B) \to \mathcal{L}(B'), \qquad \mathcal{D}_A^{(x)}\!:\mathcal{L}(A \otimes E_A) \to \mathcal{L}(A')$$

such that

$$\mathcal{C} = \sum_{x \in X} \mathcal{C}_B^{(x)} \circ \mathcal{C}_A^{(x)}(\rho_{E_A E_B}) = \sum_{x \in X} \mathcal{D}_A^{(x)} \circ \mathcal{D}_B^{(x)}(\rho_{E_A E_B}), \qquad (5)$$

namely, $\mathcal{C}$ has the two equivalent circuit realizations

$$(6)$$

$$(7)$$
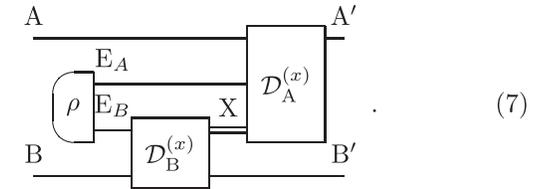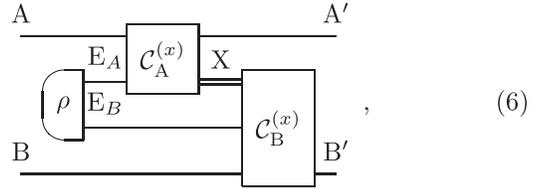
Proof.—Proof of (1) $\Rightarrow$ (2). $\mathcal{C}$ is $B \nrightarrow A'$ no signaling, therefore it can be realized as in Eq. (2), where E' is a d'-dimensional system. This system can be teleported using the entangled state $\frac{1}{\sqrt{d'}}|I\rangle\rangle$ of system $E_A' \otimes E_B'$, the Bell measurement $|B_x\rangle\rangle$ on systems E' and E_A', and classical communication of the outcome x followed by a controlled unitary U_x on system E_B', corresponding to the circuit


$$(8)$$
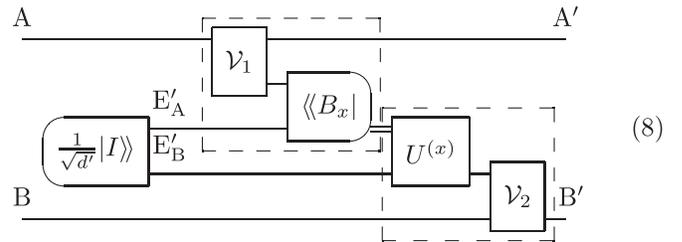
(the double wire represents the classical communication of the outcome x of the measurement).

The quantum operation $\mathcal{C}_A^{(x)}$ and the channel $\mathcal{C}_B^{(x)}$ are the grouped circuital elements in Eq. (8), and are

$$\mathcal{C}_A^{(x)}(\rho) := \langle\langle B_x|(\mathcal{V}_1 \otimes I_{E'_A})(\rho)|B_x\rangle\rangle$$

$$\mathcal{C}_B^{(x)}(\rho) := \mathcal{V}_2((U^{(x)} \otimes I_B)\rho(U^{(x)} \otimes I_B)^\dagger). \quad (9)$$

The final circuit is thus

$$(10)$$
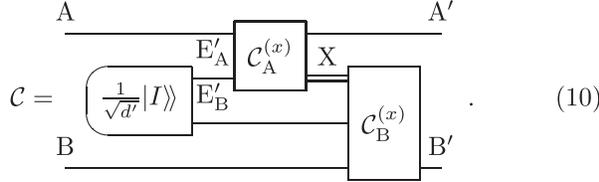
Since the channel $\mathcal{C}$ is also $A \not\rightarrow B'$ no signaling, the same argument gives:
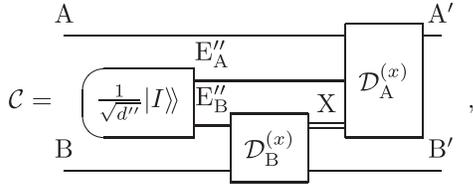
with $\mathcal{D}_A^{(x)}$ and $\mathcal{D}_B^{(x)}$ given by

$$\mathcal{D}_B^{(x)}(\rho) := \langle\langle B_x|(\mathcal{W}_1 \otimes I_{E''_B})(\rho)|B_x\rangle\rangle$$

$$\mathcal{D}_A^{(x)}(\rho) := \mathcal{W}_2((U^{(x)} \otimes I_A)\rho(U^{(x)} \otimes I_A)^\dagger). \quad (11)$$

We obtain the statement by defining E_A and E_B as d-dimensional systems, where $d := \max\{d', d''\}$, and embedding E'_J and E''_J in E_J, for $J = A, B$.

Proof of (2) $\Rightarrow$ (1).—Suppose that $\mathcal{C}$ admits the realization in Eq. (6). We can group E_B and X in the composite system E'. Then $\mathcal{C}$ is also of the form of Eq. (2), thus being $B \not\rightarrow A'$ no signaling, as proved in Refs. [7,10]. In the same way, exploiting the second scheme in Eq. (7), one can prove that $\mathcal{C}$ is also $A \not\rightarrow B'$ no signaling. ∎

Theorem 1 shows that the most general no-signaling channel differs from a localizable channel because it also admits a single round of classical communication, with the constraint that it must be possible to implement the channel exploiting communication in either direction.

We now provide a counterexample to Conjecture 1, in terms of a no-signaling channel that is atomic (i.e., it cannot be written as a convex combination of different channels whence also of no-signaling channels) and that is neither entanglement breaking nor localizable. Let A, B, X_A, X_B, W_A, W_B be qubits. We define the channel $\mathcal{R}_\alpha$ depending on α, $0 \leq \alpha \leq 1$:

$$(12)$$
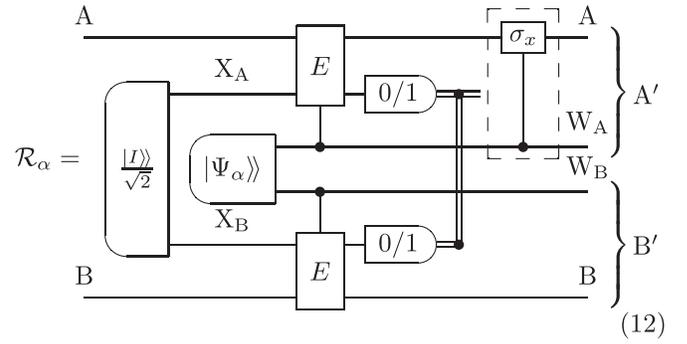
where E is the swap operator, $|\Psi_\alpha\rangle\rangle := \sqrt{\alpha}|0\rangle|0\rangle + \sqrt{1-\alpha}|1\rangle|1\rangle$, the two-qubit gate in the dashed box is a controlled-σ_x given by $\Sigma_{AW_A} := |1\rangle\langle 1|_{W_A} \otimes (\sigma_x)_A + |0\rangle\times\langle 0|_{W_A} \otimes I_A$ classically controlled by the outcomes of the measurements on the computational basis (represented by the circuital element $-\boxed{0/1}$). Notice that the classical control works as a logical AND, implying that the box Σ_{AW_A} is performed if and only if both outcomes of the measurements $-\boxed{0/1}$ are equal to 1.

We notice that circuit $\mathcal{R}_\alpha$ in Eq. (12) is implemented using local operations, entanglement, and one round of classical communication from Bob to Alice, thus being of the form of Eq. (7). One can verify that $\mathcal{R}_\alpha$ can be equivalently realized applying the controlled σ_x on systems B and W_B as follows:

$$(13)$$
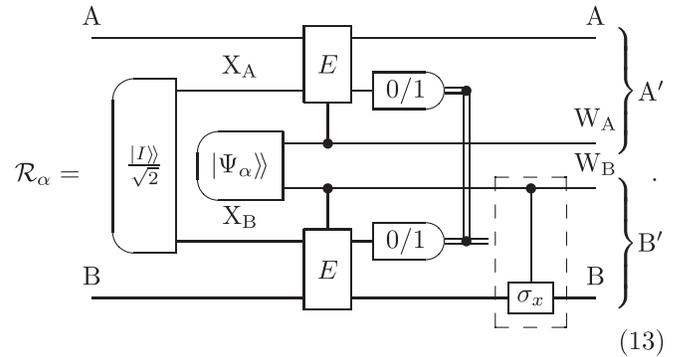
Consequently $\mathcal{R}_\alpha$ also admits a realization of the form given in Eq. (6). By Theorem 1, we can conclude that this is a no-signaling channel. The Choi-Jamiołkowski operator of $\mathcal{R}_\alpha$ is

$$R_\alpha = \sum_{m,n=0}^{1} |K_{mn}^\alpha\rangle\rangle\langle\langle K_{mn}^\alpha|,$$

$$|K_{mn}^\alpha\rangle\rangle = [(\Sigma_{AW_A})^{mn} \otimes \langle m|_{X_A}\langle n|_{X_B}]|\Phi_\alpha\rangle\rangle,$$

$$|\Phi_\alpha\rangle\rangle = (\mathbb{E} \otimes I_{AB})\Big(|I\rangle\rangle_{AB,AB} \otimes \frac{1}{\sqrt{2}}|I\rangle\rangle_{X_A X_B} \otimes |\Psi_\alpha\rangle\rangle\Big),$$

$$(14)$$

$\mathbb{E}$ denoting the tensor product of the two C swaps.

Using MATHEMATICA, we prove that $\mathcal{R}_{\tilde{\alpha}}$ with $\tilde{\alpha} := 1/6$ is a counterexample by showing that it satisfies the

following properties: (i) it is not entanglement breaking, (ii) it is not localizable, and (iii) it is atomic.

Proof of (1).—($\mathcal{R}_{\tilde{\alpha}}$ is not entanglement breaking.) A channel is entanglement breaking if and only if the corresponding Choi-Jamiołkowski operator is separable. Thus, we can prove that $\mathcal{R}_{\tilde{\alpha}}$ is not entanglement breaking by showing that $R_{\tilde{\alpha}}$ violates the Peres-Horodecki criterion for separability [11,12]. According to the criterion, if a state is separable it has a positive definite partial transpose. Numerically, one can check that $R_{\tilde{\alpha}}$ has a partial transpose with negative eigenvalues, whence we conclude that it is entangled and $\mathcal{R}_{\tilde{\alpha}}$ is not entanglement breaking.

Proof of (2).—($\mathcal{R}_{\tilde{\alpha}}$ is not localizable.) If $\mathcal{R}_{\alpha}$ were localizable [see Eq. (4)], the following observables A_n, B_m

$$ \tag{15} $$
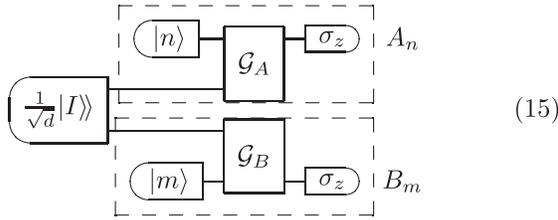
(—$\boxed{\sigma_z}$ is the measurement of σ_z) would verify the Cirel'son bound [5]: $c_\alpha := |\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle| \leq 2\sqrt{2}$. We have that $\langle A_n B_m \rangle = \mathrm{Tr}[(\sigma_A^z \otimes |n\rangle \times \langle n|_A \otimes \sigma_B^z \otimes |m\rangle\langle m|_B \otimes I_{W_A W_B})R_\alpha]$ whence [using expression in Eq. (14) for R_α] one finds $c_\alpha = |4 - 6\alpha|$. Since $c_{\tilde{\alpha}} = 3 > 2\sqrt{2}$, the Cirel'son bound is violated and $\mathcal{R}_{\tilde{\alpha}}$ cannot be localizable.

Proof of (3).—($\mathcal{R}_{\tilde{\alpha}}$ is extremal.) One can check that the matrices $\{K_{mn}^{\tilde{\alpha}\dagger} K_{m'n'}^{\tilde{\alpha}}\}$ are linearly independent. By Choi's theorem [13] the channel $\mathcal{R}_{\tilde{\alpha}}$ is extremal.

For a multipartite channel satisfying two different no-signaling conditions, an analog of Theorem 1 holds. In fact, let us consider a channel $\mathcal{C}$ with input systems labeled by a set of indices $\mathcal{J}$ and output systems labeled by a set $\mathcal{O}$. Suppose that $\mathcal{C}$ satisfies

$$ \mathrm{Tr}_{\mathcal{O}'}[R_\mathcal{C}] = I_{\mathcal{J}'} \otimes S_{\overline{\mathcal{O}'} \cup \overline{I'}}, \quad \mathrm{Tr}_{\mathcal{O}''}[R_\mathcal{C}] = I_{\mathcal{J}''} \otimes T_{\overline{\mathcal{O}''} \cup \overline{I''}}, \tag{16} $$

for certain subsets $\mathcal{J}', \mathcal{J}'' \subseteq \mathcal{J}$ and $\mathcal{O}', \mathcal{O}'' \subseteq \mathcal{O}$, where $\bar{\mathcal{S}}$ represents the set complement of $\mathcal{S}$, and for suitable Choi-Jamiołkowki operators S and T. Following the proof of Theorem 1 we can show that two circuits realizing $\mathcal{C}$ are

$$ \tag{17} $$
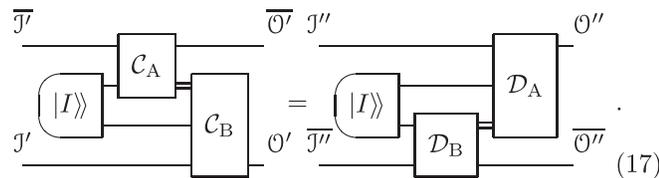
In general the subsets $\mathcal{J}', \mathcal{J}''$ are not a partition of $\mathcal{J}$. In this case we have that the circuits cannot be realized

partitioning the systems between the two local parties A and B. In particular, the input systems in $\overline{\mathcal{J}'} \cap \overline{\mathcal{J}''}$ are always assigned to the party which sends the classical message, and input systems in $\mathcal{J}' \cap \mathcal{J}''$ are assigned to the party which receives the classical message (and similarly for output systems). One can also consider more complex scenarios, i.e., channels with more than two no-signaling conditions of the kind in Eq. (16), or channels with nested conditions, for example, when the Choi-Jamiołkowski operators S and T in Eq. (16) satisfy no-signaling conditions on their own. However, the analysis of these cases is complicated, and is left as on open problem.

In conclusion, we have provided the general realization scheme of no-signaling channels, giving a counterexample to the conjecture of Ref. [8], stating that such channels are mixtures of entanglement-breaking and localizable channels. The general scheme allows for more flexibility of entanglement monogamy, opening the new problem of determining the trade-off between generated correlations and preserved entanglement. The general realization scheme looks counterintuitive, due to the presence of classical communication. However, the nontrivial constraint is the fact that an equivalent scheme must exist, with communication in the reverse direction, and remarkably this suffices to make the channel no signaling.

[1] J. Pearl, *Causality* (Cambridge University Press, Cambridge, 2009).
[2] N. Herbert, Found. Phys. **12**, 1171 (1982).
[3] A. Einstein, *The Born-Einstein Letters* (Walker, New York, 1971).
[4] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **81**, 062348 (2010).
[5] D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, Phys. Rev. A **64**, 052309 (2001).
[6] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).
[7] T. Eggeling, D. Schlingemann, and R. F. Werner, Europhys. Lett. **57**, 782 (2002).
[8] M. Piani, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **74**, 012305 (2006).
[9] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).
[10] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **80**, 022339 (2009).
[11] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[13] M. D. Choi, Linear Algebra Appl. **10**, 285 (1975).