

## Quantum cloning by cellular automata

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(Received 20 December 2012; published 29 March 2013)

We introduce a quantum cellular automaton that achieves approximate phase-covariant cloning of qubits. The automaton is optimized for  $1 \rightarrow 2N$  economical cloning. The use of the automaton for cloning allows us to exploit different foliations for improving the performance with given resources.

DOI: [10.1103/PhysRevA.87.032337](https://doi.org/10.1103/PhysRevA.87.032337)

PACS number(s): 03.67.Ac, 03.65.Aa, 03.67.Lx

### I. INTRODUCTION

Quantum cellular automata (QCAs) have attracted considerable interest in recent years [1,2] due to their versatility in tackling several problems in quantum physics. Quantum automata describing single particles correspond to the so-called quantum random walks [3], whose probability distributions can be simulated with an optical setup [4,5]. Solid-state and atom-optics systems, such as spin chains, optical lattices, or ion chains, can be viewed as implementations of QCAs, although in a Hamiltonian description. Recently, QCAs have also been considered as a model of quantum field theory at the Planck scale [6,7].

In this scenario, general coordinate transformations correspond to foliations, such as those introduced in Ref. [8] for operational structures, e.g., the digital equivalent of the relativistic boost is given by a uniform foliation over the automaton [9]. Besides the link with fundamental research, the possibility of foliations makes the QCA particularly interesting also for implementing quantum information tasks. In this paper we will explore such a potentiality for the case of quantum cloning as a sample protocol. We will show how economical cloning can be implemented by a quantum automaton and how the foliations can be optimized and exploited for improving the efficiency of the protocol.

The work is organized as follows. In Sec. II we remind some concepts related to either phase-covariant cloning and QCAs, in Sec. III we report and explain the main achieved results concerning quantum cloning by QCAs, and we eventually summarized them in Sec. IV.

### II. PRELIMINARIES

It is well known that quantum cloning of nonorthogonal states violates unitarity [10] or linearity [11] of quantum theory, and it is equivalent to the impossibility of measuring the wave function of a single system [12]. However, one can achieve quantum cloning approximately for a given prior distribution over input quantum states. For uniform Haar distribution of pure states the optimal protocol has been derived in Ref. [13], whereas for equatorial states it has been given in Refs. [14,15]. In the present paper we consider specifically this second protocol, corresponding to the clone

of the two-dimensional equatorial states:

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle). \quad (1)$$

The cloning is phase covariant in the sense that its performance is independent of  $\phi$ ; i.e., the fidelity is the same for all states  $|\phi\rangle$ . For certain numbers of input and output copies it was shown that the optimal fidelity can be achieved by a transformation acting only on the input and blank qubits, without extra ancillae [16,17]. Since these transformations act only on the minimal number of qubits, they are called “economical.” The unitary operation  $U_{pcc}$  realizing the optimal  $1 \rightarrow 2$  economical phase-covariant cloning is given by [16]

$$\begin{aligned} U_{pcc}|0\rangle|0\rangle &= |0\rangle|0\rangle, \\ U_{pcc}|1\rangle|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle), \end{aligned} \quad (2)$$

where the first qubit is the one we want to clone, while the second is the blank qubit initialized to input state  $|0\rangle$ . In Ref. [17] the economical map performing the optimal  $N \rightarrow M$  phase-covariant cloning for equatorial states of dimension  $d$  is explicitly derived for  $M = kd + N$  with integer  $k$ .

In order to analyse a QCA implementation of the economical quantum cloning, we now recall the reader some properties of QCAs we are considering here. Our automaton is one-dimensional, and a single time step corresponds to a unitary shift-invariant transformation achieved by two arrays of identical two-qubit gates in the two-layer Margolus scheme [1] reported in Fig. 1. Notice that this is the most general one-dimensional automaton with next-nearest-neighbor interacting minimal cells. Due to the locality of interactions, information about a qubit cannot be transmitted faster than two systems per time step, and this corresponds to the cell (qubit) “light cone” made of cells that are causally connected to the first. No event outside the cone can be influenced by what happened in the first cell; thus the quantum computation of the evolution of localized qubits is finite for finite numbers of time steps.

We now recall the concept of *foliation* on the gate structure of the QCA [9]. Usually, in a quantum circuit, drawn from the bottom to the top as the direction of input-output, one considers all gates with the same horizontal coordinate as simultaneous transformations. A foliation on the circuit corresponds to stretching the wires (namely, without changing the connections) and considering as simultaneous all the gates that lie on the same horizontal line after the stretching. Such a horizontal line can be regarded as a *leaf* of the foliation on the circuit before the stretching transformation.

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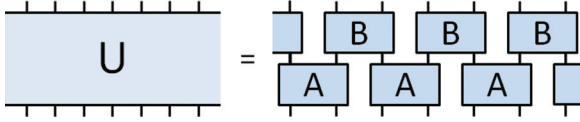


FIG. 1. (Color online) Realization of a one-dimensional quantum cellular automaton with a structure composed of two layers of gates,  $A$  and  $B$ . This is the most general one-dimensional automaton with next-nearest-neighbor interacting minimal cells.

Therefore, a foliation corresponds to a specific choice of simultaneity of transformations (the “events”); namely, it represents an observer or a reference frame. Examples of different foliations are given in Fig. 2. Upon considering the quantum state at a specific leaf as the state at a given time (at the output of simultaneous gates), different foliations correspond to different state evolutions achieved with the same circuit. Therefore, in practice we can achieve a specific state belonging to one of the different evolutions by simply cutting the circuit along a leaf and tapping the quantum state from the resulting output wires (the operation of “stretching” wires should be achieved by remembering that by convention the wires represent identical evolutions, not “free” evolutions).

### III. PHASE-COINVARIANT CLONING BY QCAs

We will now show how to perform a  $1 \rightarrow 2N$  phase-covariant cloning of the equatorial states (1) with a QCA of  $N$  layers, with all gates identical, performing the unitary transformation denoted by  $A$ , acting on two qubits. Due to causality, we can restrict our treatment to the light cone centered in the state to be cloned  $|\phi\rangle$  and initialize all blank qubits to  $|0\rangle$ , as shown in Fig. 3. By requiring phase covariance for the cloning transformation, the unitary operator  $A$  must commute with every transformation of the form  $P_\chi \otimes P_\chi$ , where  $P_\chi$  is the general phase-shift operator  $P_\chi = \exp[\frac{i}{2}(\mathbf{1} - \sigma_z)\chi]$  for a single qubit, with  $\sigma_z$  being the Pauli matrix along  $z$ . Therefore, we impose the condition

$$[A, P_\chi \otimes P_\chi] = 0, \forall \chi. \quad (3)$$

This implies that the matrix  $A$  must be of the form  $A = \text{diag}(1, V, 1)$ , where  $V$  is a  $2 \times 2$  unitary matrix. Notice that the transformation  $A$  then acts nontrivially only on the subspace

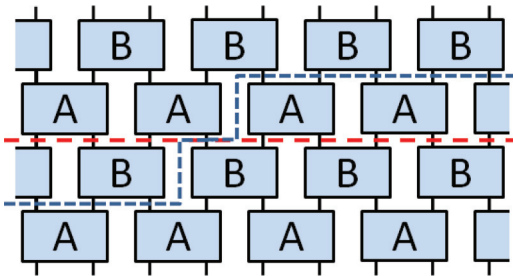


FIG. 2. (Color online) Foliation over the automaton. Two leaves of two different uniform foliations are depicted with dashed lines in different colors (the complete foliation is obtained upon repeating the leaf vertically). The systems along each leaf are taken as simultaneous. The red “cut” is usually referred to as the rest-frame foliation.

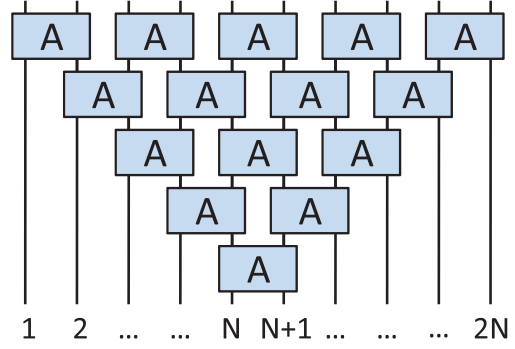


FIG. 3. (Color online) Cone of gates which contribute to the phase-covariant cloning, given the input state  $|\phi\rangle$  at site  $N$ .

spanned by the two states  $\{|01\rangle, |10\rangle\}$ , and it is completely specified by fixing  $V$ .

In order to derive the optimal cloning transformation based on this kind of QCA we will now maximize the average single-site fidelity of the  $2N$ -qubit output state with respect to the unitary operator  $A$ . In order to achieve this, we write the initial state of  $2N$  qubits in the following compact form:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\Omega\rangle + e^{i\phi}|N\rangle), \quad (4)$$

where we define  $|\Omega\rangle = |0 \dots 0\rangle$  as the “vacuum state” with all qubits in the state down and  $|k\rangle = |0 \dots 01_k 0 \dots 0\rangle$  as the state with the qubit up in the position  $k$  and all other qubits in the state down. Without loss of generality in the above notation the qubit to be cloned is supposed to be placed at position  $N$ , and it is initially in the state  $|\phi\rangle$ . Since gate  $A$  preserves the number of qubits up [18], the evolved state through each layer will belong to the Hilbert space spanned by the vacuum state and the  $2N$  states with one qubit up. The whole dynamics of the QCA can then be fully described in a Hilbert space of dimension  $2N + 1$ . The output state can thus be generally written as

$$|\Psi_{2N}\rangle = \frac{1}{\sqrt{2}} \left( |\Omega\rangle + e^{i\phi} \sum_{k=1}^{2N} \alpha_k |k\rangle \right), \quad (5)$$

where the amplitudes  $\alpha_k$  of the excited states depend only on the explicit form of the gate  $A$ .

The reduced density matrix  $\rho_k$  of the qubit at site  $k$  can then be derived from the output state (5) as

$$\begin{aligned} \rho_k &= \text{Tr}_{\bar{k}}[|\Psi_{2N}\rangle\langle\Psi_{2N}|] \\ &= \frac{1}{2} \left[ \left( 1 + \sum_{j \neq k} |\alpha_j|^2 \right) |0\rangle\langle 0| + e^{-i\phi} \alpha_k^* |0\rangle\langle 1| \right. \\ &\quad \left. + e^{i\phi} \alpha_k |1\rangle\langle 0| + |\alpha_k|^2 |1\rangle\langle 1| \right], \end{aligned} \quad (6)$$

where  $\text{Tr}_{\bar{k}}$  denotes the trace on all qubits except qubit  $k$ . The local fidelity of the qubit at site  $k$  with respect to the input state  $|\phi\rangle$  then takes the simple form

$$F_k = \langle\phi|\rho_k|\phi\rangle = \frac{1}{2}(1 + \text{Re}\{\alpha_k\}). \quad (7)$$

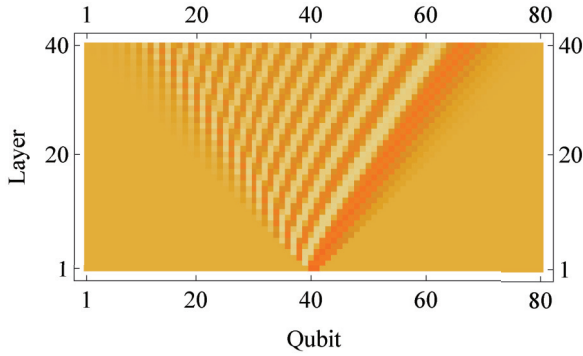


FIG. 4. (Color online) Chromatic map of the local fidelities in terms of the considered qubit and the layer. The orange color is brighter for increasing local fidelity. The simulation involves a number of layers  $N = 40$ , while the total number of qubits is  $2N$  since it doubles at each layer.

As we can see,  $F_k$  depends only on the amplitude  $\alpha_k$  of the state with a single qubit up exactly at  $k$ . Since gate  $A$  is generally not invariant under an exchange of the two qubits, the fidelities at different sites will be, in general, different. We will then consider the average fidelity  $\bar{F} = \frac{1}{2N} \sum_{k=1}^{2N} F_k$  as a figure of merit to evaluate the performance of the phase-covariant cloning implemented by QCA. Notice that the whole procedure corresponds to a unitary transformation on the  $2N$ -qubit system, without introducing auxiliary systems; namely, it is an economical cloning transformation.

The calculation of the amplitudes  $\alpha_k$  was performed numerically by updating at each layer the coefficients of state (5). Notice that the amplitude of layer  $j$  and site  $k$  influences only the amplitudes of the subsequent layer  $j + 1$  and either sites  $k - 1, k$  or  $k, k + 1$ , depending on whether state  $|1\rangle$  enters the right or left wire of  $A$ . The action of  $A$  on qubits  $j$  and  $j + 1$  is given by

$$A(j, j + 1)|k\rangle = \begin{cases} v_{22}|j\rangle + v_{12}|j + 1\rangle & \text{if } k = j, \\ v_{21}|j\rangle + v_{11}|j + 1\rangle & \text{if } k = j + 1, \\ |k\rangle & \text{otherwise,} \end{cases} \quad (8)$$

where  $v_{ij}$  are the entries of operator  $V$  in the basis  $\{|01\rangle, |10\rangle\}$ . Notice that the vacuum state  $|\Omega\rangle$  is invariant under the action of  $A$ . The iteration of Eq. (8) for each layer then leads to the amplitudes of output state (5).

#### A. Performances in the rest frame

As a first explicit example we will consider a QCA employing the optimal  $1 \rightarrow 2$  phase-covariant cloning (2). In this case gate  $A$  must implement the unitary transformation (2). The nontrivial part  $V$  of gate  $A$  can then be chosen to be

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (9)$$

where all coefficients are real. The corresponding local fidelities at every layer are reported in Fig. 4. As we can see, Fig. 4 exhibits fringes of light and dark color. Moreover, the light cone defined by causality can be clearly seen: outside this cone no information about the initial state can arrive; thus every system has the same fidelity of  $1/2$ . Notice that there is a

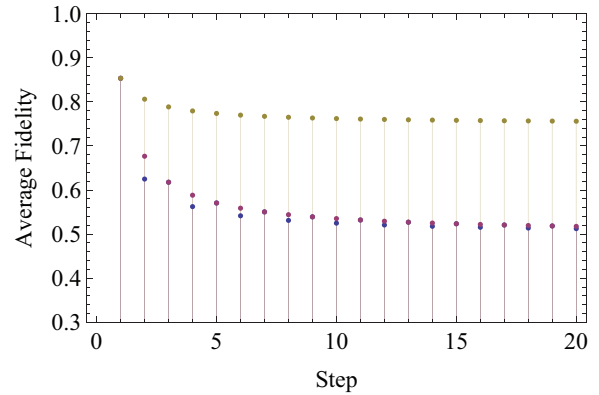


FIG. 5. (Color online) The average  $1 \rightarrow 2N$  phase-covariant cloning fidelity achieved by the QCA in the rest frame (see Fig. 2). Comparison with the optimal economical phase-covariant cloning in Ref. [15]. The purple dots represent the QCA cloning optimized over the unitary gate  $A$ . The blue lower dots correspond to the use of gate  $A$  to achieve the optimal  $1 \rightarrow 2$  cloning  $U_{pcc}$  in Eq. (2). The yellow upper dots represent the unrestricted optimal economical cloning.

sort of line, approaching the right top corner, along which the fidelity is quite high. Regarding the local fidelities of the final states, they are, in general, quite different from each other and can vary very quickly even between two neighboring qubits. The average fidelity is reported in Fig. 5 as a function of the number of layers. Notice that the average fidelity of the optimal economical phase-covariant quantum cloning (without the constraint of automaton structure) approaches the value  $3/4$  for a large number of output copies [15].

In order to improve the average fidelity we then maximized it with respect to the four parameters defining the unitary operator  $V$ . Numerical results achieved up to  $N = 20$  show that the optimal cloning performed in this case is not much better than the one given by the iteration of (9). Eventually, the latter turns out to be outperformed only when the number of layers composing the automaton is odd, as shown in Fig. 5.

Further numerical results show that no gain can be achieved if the automaton is composed of layers of two different gates  $A$  and  $B$ . Actually, in this case it surprisingly turns out that the optimal choice corresponds to  $B = A$ ; namely, we do not exceed the average fidelity obtained by employing a single type of gate. As a result, since all one-dimensional QCA with next-nearest-neighbor interacting cells with two qubits can be implemented by a two-layered structure, we have then derived the optimal phase-covariant cloning transformation achievable by the minimal one-dimensional QCA.

#### B. Performances exploiting different foliations

We will now show that the average fidelity in the case of a single-gate automaton can be improved by considering different foliations. Suppose that we are given a fixed number  $M$  of identical gates  $A$  to implement a QCA. We are then allowed to place the gates in any way such that the causal structure of the considered automaton is not violated. What configuration, i.e., the foliation, performs the optimal phase-covariant cloning for fixed  $M$ ? In this framework we have to maximize not only over the parameters that define  $V$  but also

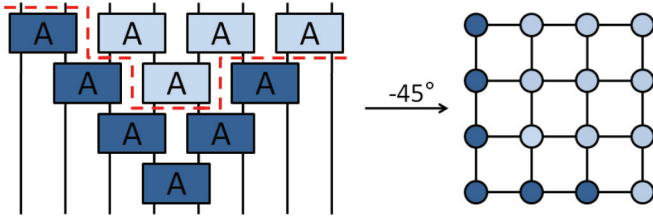


FIG. 6. (Color online) Illustration of the classification of foliations. A possible foliation with  $M = 6$  gates is given. From the correspondence between the gates lying under the cut and the rotated dots on the right, we identify this foliation with the partition  $\{4, 1, 1\}$ .

over all possible foliations. Thus, the  $M$  fixed gates play the role of computational resources, and the optimality is then defined in terms of both the parameters characterizing the single gate  $A$  and the disposition of the gates in the network. As a first example, suppose that we are given  $M = 3$  gates. In this case there are three inequivalent foliations: one for the rest frame (see Fig. 2) and two along the straight lines defining the light cone. As expected, for increasing  $M$  the counting of foliations becomes more complicated, and the problem is how to choose and efficiently investigate each possible foliation. It turns out that the problem of identifying all possible foliations of a QCA of the form illustrated in Fig. 3 for a fixed number of gates  $M$  is related to the partitions of the integer number  $M$  itself (by partition we mean a way of writing  $M$  as a sum of positive integers, a well known concept in number theory [19]). Two sums that differ only in the order of their addends are considered to be the same partition. For instance, the partitions of  $M = 3$  are exactly 3 and are given by  $\{3\}$ ,  $\{2, 1\}$ , and  $\{1, 1, 1\}$ , while the partitions of  $M = 6$ , corresponding to a three-layer setting in the rest frame, are 11 and are given by  $\{6\}$ ,  $\{5, 1\}$ ,  $\{4, 2\}$ ,  $\{4, 1, 1\}$ ,  $\{3, 3\}$ ,  $\{3, 2, 1\}$ ,  $\{3, 1, 1, 1\}$ ,  $\{2, 2, 2\}$ ,  $\{2, 2, 1, 1\}$ ,  $\{2, 1, 1, 1, 1\}$ , and  $\{1, 1, 1, 1, 1, 1\}$ . The link between foliations and partitions is illustrated in Fig. 6, which shows how partitions can be exploited to identify foliations. For a fixed number of gates  $M$  the number of foliations is then automatically fixed, and each foliation corresponds to a single partition. The correspondence is obtained as follows. Each addend represents the number of gates along parallel diagonal lines, starting from the vertex of the light cone, as shown in Fig. 6 for the particular case of  $M = 6$ .

TABLE I. Results of the maximization over foliations up to  $M = 28$ , i.e., QCA composed of up to seven layers.

$M(\text{Layers})$	$\bar{F}_{\text{rest}}$	$\bar{F}$	Optimal foliation
1 (1)	0.853	0.853	{1}
3 (2)	0.676	0.693	{3}
6 (3)	0.617	0.679	{2,2,2}
10 (4)	0.588	0.670	{4,3,3}
15 (5)	0.570	0.653	{4,4,4,3}
21 (6)	0.558	0.614	{4,3,2,2,2,2,2,2}
28 (7)	0.550	0.603	{6,6,6,5,5}

Based on this correspondence, we can investigate the performance of the phase-covariant cloning as follows. For any fixed foliation, we first maximize the average fidelity with respect to the four parameters of the unitary  $V$ , defining gate  $A$ . Then we choose the highest average fidelity that we have obtained by varying the foliation. We worked out this procedure numerically for  $M = 1, 3, 6, 10, 15, 21, 28$ , i.e., the number of gates composing the QCA with  $N = 1, 2, 3, 4, 5, 6, 7$  layers, respectively. Our results are shown in Table I, where the maximization in the rest frame is also reported for comparison. As we can see, exploiting different foliations leads to a substantial improvement of the average fidelity.

#### IV. CONCLUSIONS

In summary, we have introduced a way of achieving quantum cloning through QCAs. We have derived the optimal automaton achieving economical phase-covariant cloning for qubits. We have shown how the fidelity of cloning can be improved by varying the foliation over the QCA, with a fixed total number of gates used. By developing an efficient method to identify and classify foliations by means of number theory, we have optimized the performance of the QCA phase-covariant cloning for a given fixed number of gates and have obtained in this way the most efficient foliation.

#### ACKNOWLEDGMENTS

M.R. gratefully acknowledges an enlightening discussion with Davide A. Costanzo.

[1] P. Arrighi, V. Nesme, and R. F. Werner, in *Language and Automata Theory and Applications*, Lect. Notes Comput. Sci., Vol. 5196 (Springer, Berlin, 2008), p. 64.  
[2] P. Arrighi, V. Nesme, and R. F. Werner, *J. Comput. Syst. Sci.* **77**, 372 (2011).  
[3] J. Kempe, *Contemp. Phys.* **44**, 307 (2003).  
[4] A. Schreiber *et al.*, *Science* **336**, 55 (2012).  
[5] L. Sansoni, F. Sciarrino, G. Vallone, P. Mataloni, A. Crespi, R. Ramponi, and R. Osellame, *Phys. Rev. Lett.* **108**, 010502 (2012).  
[6] G. M. D'Ariano, *Phys. Lett. A* **376**, 697 (2012).

[7] A. Bisio, G. M. D'Ariano, and A. Tosini, arXiv:1212.2839.  
[8] L. Hardy, in *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, edited by H. Halvorson (Cambridge University Press, New York, 2011), pp. 409–442.  
[9] G. M. D'Ariano, in *Quantum Theory: Reconsideration of Foundations—5*, edited by A. Y. Khrennikov, AIP Conf. Proc. No. 1232 (AIP, Melville, NY, 2010), p. 3.  
[10] H. P. Yuen, *Phys. Lett. A* **113**, 405 (1986).  
[11] W. K. Wootters and W. H. Zurek, *Nature (London)* **299**, 802 (1982).

- [12] G. M. D'Ariano and H. P. Yuen, *Phys. Rev. Lett.* **76**, 2832 (1996).
- [13] R. F. Werner, *Phys. Rev. A* **58**, 1827 (1998).
- [14] D. Bruß, M. Cinchetti, G. M. D'Ariano, and C. Macchiavello, *Phys. Rev. A* **62**, 012302 (2000).
- [15] P. C. Richter, *Phys. Rev. A* **76**, 042306 (2007).
- [16] T. Durt, J. Fiurášek, and N. J. Cerf, *Phys. Rev. A* **72**, 052322 (2005).
- [17] F. Buscemi, G. M. D'Ariano, and C. Macchiavello, *Phys. Rev. A* **71**, 042327 (2005).
- [18] It is easy to show that, since  $[P_\chi \otimes P_\chi, \sum_k \sigma_{z,k}] = 0$  for all  $\chi$ , gate  $A$  commutes with the operator  $\sum_k \sigma_{z,k}$ , and thus the number of qubits up is preserved during the evolution.
- [19] G. E. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, 1976).