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Universal quantum observables

Giacomo Mauro D'Ariano [a,b]

[a] *Quantum Optics and Information Group, Istituto Nazionale di Fisica della Materia, Unità di Pavia, Italy*
[b] *Department of Electrical and Computer Engineering, Northwestern University, Evanston, IL 60208, USA*

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Abstract

It is shown how one can estimate the ensemble average of all operators of a quantum system by measuring only one fixed "universal" observable on an extended Hilbert space. This is equivalent to run a tomographic reconstruction in a kind of "quantum parallelism", measuring all the quorum observables with a single universal observable. An experimental implementation in quantum optics is given, based on Kerr cross-phase modulated homodyning. © 2002 Elsevier Science B.V. All rights reserved.

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Quantum tomography [1–5] is a method for estimating the ensemble average of all operators of a quantum system—including its density matrix—from a set of measurements of a *quorum* of observables, i.e., a "complete" set of noncommuting observables. The name *quantum tomography* originated in quantum optics, where the set of quadrature probability distributions for varying phase was recognized [6] as the Radon transform of the Wigner function, the Radon transform being the basic imaging tool in computerized medical tomography. Such analogy gave the name to a first qualitative technique for measuring the matrix elements of the radiation density operator [7]. A first quantitative method was given in Ref. [8] (for a review, see Refs. [2,4]), and is currently used in quantum optical labs [9–13]. The method has been then generalized to the estimation of an arbitrary observable of the field [14] and to arbitrary quantum system [15–18]. Finally, very recently, a method for tomographic estimation of the unknown quantum operation [19] of a quantum device has been presented [20], exploiting the "quantum parallelism" of an entangled input state which plays the role of a "superposition of all possible input states".

In this Letter I will show how another kind of quantum parallelism can be exploited in order to run the whole tomographic process in parallel, by measuring all the quorum observables in a single *universal observable*. In analogy with the tomographic Radon-transform reconstruction of a two-dimensional image [2–4] this method would resemble a kind of "quantum holography", with the whole Radon-transform included in the single universal observable.

After briefly reviewing the general tomographic estimation approach, I will outline the construction of

E-mail address: dariano@unipv.it (G.M. D'Ariano).
URL address: http://www.qubit.it.

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universal observables, characterizing all of them, and giving a way for deconvolving measurement noise. I will also sketch an experimental implementation in quantum optics, based on Kerr cross-phase modulation and homodyning.

Let us now recall the general quantum tomographic method. In the following, I will use a more convenient definition of observable than the usual selfadjoint operator. I will call *observable* any complex operator O that corresponds to a complete orthonormal resolution of the identity $I = \sum_{\vec{s}} |\vec{s}\rangle\langle\vec{s}|$, where $\vec{s} = (s_1, \ldots, s_n)$ denotes (a set of generally complex) "eigenvalues". In this way, all observables are in one-to-one correspondence with *normal operators* O, i.e., operators commuting with their adjoint, as $[O, O^\dagger] = 0$, and any complex function F of an observable will be itself observable, F denoting the normal operator $F = \sum_{\vec{s}} F(\vec{s})|\vec{s}\rangle\langle\vec{s}|$ that resorts to the same physical measurement, but with a different data processing [21].

As already mentioned, the idea of quantum tomography is the possibility of estimating the ensemble average of all operators of a quantum system from a set of measurements of a *quorum* of observables, which in the following I will denote by $\{c(l)\}$. We call *quorum* any set of operators that span the linear space $\mathsf{L}(\mathsf{H})$ of operators on H. This means that any operator A can be expanded as $A = \sum_l \mathrm{Tr}[b^\dagger(l)A]c(l)$ [23], where $\{b(l)\}$ is the dual set of $\{c(l)\}$ satisfying $\mathrm{Tr}[b^\dagger(i)c(j)] = \delta_{ij}$, and we say that $\{c(l)\}$ and $\{b(l)\}$ form a biorthogonal spanning couple for $\mathsf{L}(\mathsf{H})$. Which biorthogonal couples correspond to spanning sets? They must satisfy the so-called *orthogonality relation*

$$\sum_l \langle n|b^\dagger(l)|r\rangle\langle s|c(l)|m\rangle = \delta_{nm}\delta_{rs}, \tag{1}$$

where $\{|n\rangle\}$ denotes any orthonormal basis on H. Notice that the name "orthogonality relation" for identity (1) does not mean that the spanning set itself is orthogonal, and most of the times the physical spanning sets of observables are overcomplete, and the index l of the set has to be regarded as continuous, with the sum in Eq. (1) being replaced by an integral. Notice that the continuous version $\Delta(i,j) \doteq \mathrm{Tr}[b^\dagger(i)c(j)]$ of the "Kroneker delta" is generally not a "Dirac delta": in fact, it can be bounded, and have support on the whole index manifold (such "delta" is generally referred to as "self-reproducing kernel", since it works effectively as a Dirac delta under the integral of the operator expansion). Finally, it is easy to prove that any spanning set must be *irreducible*, i.e., it must satisfy the following properties: (i) there are no proper subspaces of H which are left invariant under the action of all $c(l)$; (ii) it has a *trivial centralizer*, namely the only operators that commute with all $c(l)$ are multiple of the identity.

Given a *quorum*, i.e., a spanning set of observables, the tomographic estimation of the ensemble average $\langle A\rangle$ is simply obtained in form of double average—over both the ensemble and the quorum—of the *unbiased estimator* $\mathrm{Tr}[b^\dagger(l)A]c(l)$ with random l. The most popular example of quantum tomography is the optical *quantum homodyne tomography* [2,4], where the quorum (self-dual) is given by the operators $c(k,\phi) = \exp(ikX_\phi)$, with $X_\phi = \frac{1}{2}(a^\dagger e^{i\phi} + ae^{-i\phi})$ denoting the quadrature of the radiation field mode with annihilator a at phase ϕ with respect to the local oscillator. Notice that for estimating the density matrix also the maximum-likelihood strategy can be used instead of the averaging procedure [16,24]. Moreover, there is a general method [16] for deconvolving instrumental noise when measuring the quorum, which resorts to finding the biorthogonal basis for the "noisy" quorum: this will be used in the following. For example, in homodyne tomography deconvolution of noise from non-unit quantum efficiency is possible [2,4].

For tomography of multipartite quantum systems, when they are *distinguishable*—i.e., they can be measured separately—a quorum is simply given by the tensor product of single-system quorums, e.g., for two identical systems one has the quorum $\{c(k) \otimes c(l)\}$ with dual set $\{b(k) \otimes b(l)\}$ and expansion of any joint operator J as $J = \sum_{lk} \mathrm{Tr}[b^\dagger(k) \otimes b^\dagger(l)J]c(k) \otimes c(l)$. An example of application is the experiment of Ref. [12], which has been performed on a twin beam from parametric down-conversion of vacuum. If the systems are indistinguishable, a permutation-invariant global quorum on the tensor-product Hilbert space is needed: this is the case of multimode homodyne tomography with a single local oscillator [25], where, in principle, the full joint density matrix of a multimode radiation field can be recovered using a single homodyne detector with tunable mode-shape local oscillator.

Let us now come back to the idea of "universal observables". How can I make a complete tomographic reconstruction by measuring only a single observable? Consider the following observable H, with $[H, H^\dagger]= 0$, on the extended Hilbert space $\mathsf{H} \otimes \mathsf{K}$

$$H = \sum_l c(l) \otimes |l\rangle\langle l| \equiv c(K), \tag{2}$$

where $\{|l\rangle\} \in \mathsf{K}$ denotes an orthonormal basis in the ancillary Hilbert space K corresponding to an auxiliary observable $K = \sum_l l|l\rangle\langle l|$ (K is generally infinite-dimensional, and in the continuous case $|l\rangle$ denotes a Dirac-normalized orthogonal vector). Then, prepare the ancilla in a state σ with all nonvanishing diagonal matrix elements. In order to estimate the ensemble average of the operator A on H measure the following function F_A of the observable H

$$F_A = \sum_l \frac{\mathrm{Tr}[b^\dagger(l)A]}{\langle l|\sigma|l\rangle} c(l) \otimes |l\rangle\langle l|, \tag{3}$$

with the ancilla of the apparatus in the state σ. Et voilà: the ensemble average of the operator A is just the ensemble average of the function F_A of observable H, namely

$$\langle A\rangle \equiv \langle F_A\rangle. \tag{4}$$

Achieving the estimation of the ensemble average $\langle A\rangle$ of a particular operator A resorts only to change the data processing rule in the measurement of the "universal" observable H, which then allows to estimate every ensemble average for the quantum system on H. The general idea is synthetically sketched in Fig. 1.

It is easy to characterize all possible universal observables of the form (2). In fact, irreducibility of any spanning set of operators implies that the only observables which commute with a universal operator must be ancilla observables, namely operators of the form $I \otimes K$. At the same time, an ancillary observable K classifies a class of universal observables H_K, namely those observables that commute with $I \otimes K$, i.e., $H_K = c(K)$ for any given spanning set $\{c(l)\}$. Physically, the universal observable will be achieved by jointly measuring two observables $O_S \otimes O_A$ on system and ancilla, after an interaction U that gives $H_K \equiv c(K) = U^\dagger O_S \otimes O_A U$ for a given spanning set $\{c(l)\}$.

As an example of application, let us consider the following universal observable for quantum homodyne tomography, with the quadratures X_ϕ at different phases ϕ with respect to the local oscillator achieved

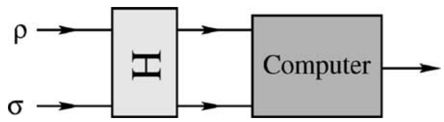

Fig. 1. General scheme for a *Universal Observable* H. By changing the data processing on the computer, one can measure the ensemble average of any operator of the quantum system with unknown density operator ρ by measuring the observable H on an extended Hilbert space with the ancilla prepared essentially in any state σ (see text). Notice that once a sample of measurement outcomes is recorded, it can be reused for any other observable, by just re-evaluating a different data processing.

via cross-Kerr interaction $U = \exp(i\kappa a^\dagger a b^\dagger b)$ as follows

$$X_{\kappa b^\dagger b} = U^\dagger X_0 U. \tag{5}$$

Comparing this case with the general scheme for universal observables, here $X_{\kappa b^\dagger b}$ plays the role of the universal observable $C(K)$ in Eq. (2), the ancillary observable O_A being just the photon number $b^\dagger b$ of the phase-modulating mode b, whereas the fixed system observable O_S is given by the quadrature X_0 at a fixed zero-reference phase, with the cross-Kerr interaction $U = \exp(i\kappa a^\dagger a b^\dagger b)$ modulating the phase ϕ of the quadrature X_ϕ. For the preparation of the ancillary mode with annihilator b, a sufficiently excited coherent state would be perfectly suitable. In fact, for irrational κ/π (in practice κ/π not too close to the fraction of two integers with one or two digits), since phases $\phi_n = \kappa n$ will be dense on the unit circle, a practically uniform phase probability distribution will be achieved for a coherent state with $\kappa\langle b^\dagger b\rangle \gg 2\pi$, thanks to the compensation term $\langle l|\sigma|l\rangle^{-1}$ in Eq. (3). The schematic apparatus for such Kerr-homodyne universal observable is given in Fig. 2. The obvious advantage of such a Kerr homodyne universal observable with respect to conventional homodyne tomography would be the automation of the quadrature phase tuning in a physical full-optical machine. Moreover, since a very large number of photons $b^\dagger b$ would be involved (that is needed due to the smallness of the Kerr coupling κ) linear photodetectors with high quantum efficiency η_D are easily available for such photon-number measurement, and the effect of nonunit quantum efficiency can be easily unbiased by just rescaling the photon-number by η_D.

Notice that although the estimation is unbiased for almost arbitrary preparation σ of the ancilla (the only

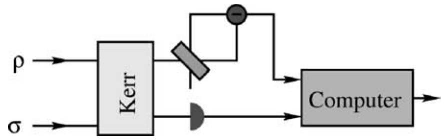

Fig. 2. Schematic apparatus for a Kerr-homodyne universal observable, as an example of application of the general method. A homodyne detector with fixed phase local oscillator is used. Instead of rotating the phase of the local oscillator as in the customary homodyne tomography, the input state ϕ of the tomographed system in the unknown state ρ is rotated via Kerr cross-phase modulation in interaction with a state σ, which could be any sufficiently excited coherent state. For a suitable Kerr coupling an essentially uniform phase probability distribution is achieved (see text).

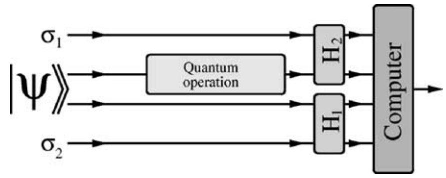

Fig. 3. Universal observable for a quantum operation (for tomography of quantum operations see Ref. [20]). An entangled input state $|\psi\rangle\rangle = \sum_{nm} \psi_{ij} |i\rangle \otimes |j\rangle$ is needed with invertible coefficients matrix $\psi = \{\psi_{ij}\}$, such as a twin-beam from parametric down-conversion. Only one of the two beams suffers the quantum operation. The apparatus measures jointly two universal observables H_1 and H_2 with two ancillas prepared in the state σ_1 and σ_2, respectively.

constraint for σ is to have nonvanishing diagonal elements in the ancilla observable), however, different choices of σ will distribute statistical errors in different ways, depending on the particular chosen spanning set $\{c(l)\}$. Moreover, as in the case of customary quantum tomography, the instrumental noise in the detection apparatus can be deconvolved by properly tuning the data processing in Eq. (3), thus making the measurement noise-unbiased. For example, a noise on the H channel would change the spanning set $c(l)$ linearly as $\mathrm{N}(c(l)) = \sum_m N_{ml} c(m)$, N denoting the quantum operation [19] of the noise. If the matrix N_{ml} is invertible, the set $\{\mathrm{N}(c(l))\}$ is still a spanning set, and the data-processing rule can be made unbiased by replacing the dual set $\{b(l)\}$ with the set $\{\mathrm{M}(b(l))\}$ with $\mathrm{M} = (\mathrm{N}^\dagger)^{-1}$. On the other hand, a noise on the ancillary channel K can be compensated by replacing the term $\langle l|\sigma|l\rangle^{-1}$ in Eq. (3) with $\langle l|\mathrm{Q}^\vee(\sigma)|l\rangle^{-1}$, when the quantum operation Q of the noise is invertible (here $\mathrm{Q}^\vee$ denotes the dual quantum operation of Q, i.e., its Schroedinger picture version). It is clear that this method can be extended also to the case of joint noise on $\mathsf{H} \otimes \mathsf{K}$ [26].

Do we have universal observables for quantum operation E [19]? The answer is obviously yes. In fact, as seen in Ref. [20], in order to estimate the matrix elements of an unknown quantum operation in some basis, one needs bipartite tomographic measurement on an entangled vector $|\psi\rangle\rangle = \sum_{nm} \psi_{ij} |i\rangle \otimes |j\rangle$ with invertible coefficients matrix $\psi = \{\psi_{ij}\}$, such as a twin-beam from parametric down-conversion, where one of the two parties of the vector has experimented the quantum operation E. In other words we just need the quantum tomography of $\mathrm{E} \otimes \mathrm{I}(|\psi\rangle\rangle\langle\langle\psi|)$. The apparatus would measure jointly two universal observables (which together make the overall universal observable), with the need of two ancillas. The two universal observables H_1 and H_2 need not to be the same, and the two ancilla need not to be identically prepared, all compensations being included in the data-processing rule. Estimation rules can be found in Ref. [20]. The general idea is synthetically sketched in Fig. 3.

In conclusion, we have seen how it is possible to perform quantum tomography of a quantum system—i.e., to estimate the ensemble average of any operator, including the matrix-form of its unknown state or of any unknown quantum operation experimented by the system—by measuring only one fixed "universal observable" on an extended Hilbert space. The ancillary part of the apparatus can be prepared essentially in any desired practical quantum state. All possible universal observables have been characterized, and a general method for compensating instrumental noise has been given. A prototype of application in quantum optics has been suggested, based on Kerr cross-phase modulated homodyning.

I hope that the present method will help in establishing soon new powerful tools for engineering more flexible quantum measurements, and for performing quantum characterization of devices and media for the future quantum information technology.

Acknowledgements

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$\eta > \frac{1}{2}$ [4]. In general, the possibility of deconvolving noise depends also on the operator A whose ensemble average $\langle A\rangle$ has to be estimated [14]. A general method for establishing the possibility of noise deconvolution will be given elsewhere [28].

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