



# On the realization of Bell observables

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## Abstract

We show how Bell observables on a bipartite quantum system can be obtained by local observables via a controlled-unitary transformation. For continuous variables this result holds for the Bell observable corresponding to the non-conventional heterodyne measurement on two radiation modes, which are connected through a 50–50 beam-splitter to two local observables given by single-mode homodyne measurements. A simple scheme for a controlled-unitary transformation of continuous variables is also presented, which needs only two squeezers, a parametric down-converter and two beam splitters.

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## 1. Introduction

The non-classical correlations of entangled quantum systems are the basis for the engineering of the next-generation devices for quantum information processing [1]. In particular, the so-called *Bell measurements* play a pivotal role in most quantum processing techniques, as they are essential in any teleportation scheme [2] or dense coding protocol [3], and recently have been proved an invaluable resource for achieving “informationally complete” measurements [4]. Nonetheless the Bell measurements still represent a serious experimental challenge, and a gen-

eral scheme for designing them would be particularly welcomed.

The name *Bell measurement* is generally designating a nonseparable joint measurement on a bipartite quantum system, typically a POVM made of rank-one operators proportional to projectors on maximally entangled vectors [5]. Here we will focus attention on the special case of the *Bell observable*, corresponding to an orthonormal basis of maximally entangled vectors. We will show how a Bell observable can be achieved by local measurements via a (nonlocal) interaction of the *controlled-unitary* form—a generalization to dimension  $d > 2$  of the controlled-NOT for qubits—corresponding to a *coherence-preserving choice* among unitary transformations controlled by the state preparation of an ancilla. This result is interesting because it emphasizes the pivotal role of the

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controlled-unitary transformation in quantum information processing. For continuous variables the same result holds for the Bell observable realized by heterodyning two radiation modes, which is connected through a 50–50 beam-splitter to two local observables describing single-mode homodyne measurements. In this way we will also see how a simple scheme for a controlled-unitary transformation of continuous variables can be achieved, by just using only two squeezers, a parametric down-converter, and two beam splitters.

After introducing some useful notation in Section 2, in Section 3 we prove the connection between Bell and local observables for finite dimension. In Section 4 we give the continuous-variable case, connecting the heterodyne Bell observable to single-mode homodyne measurements via a 50–50 beam-splitter, and presenting the optical scheme for the controlled-unitary transformation. Section 5 closes the Letter with a summary and open problems.

## 2. Some notation

We will make extensive use of the following correspondence between Hilbert–Schmidt operators on the Hilbert space  $\mathcal{H}$  and vectors in  $\mathcal{H} \otimes \mathcal{H}$  [6]

$$A = \sum_{m,n} A_{mn} |m\rangle\langle n|, \quad |A\rangle\rangle = \sum_{m,n} A_{mn} |m\rangle|n\rangle, \quad (1)$$

where the double-ket symbol  $|A\rangle\rangle$  will be used to remind the correspondence of the vector  $|A\rangle\rangle \in \mathcal{H} \otimes \mathcal{H}$  with the operator  $A$  on  $\mathcal{H}$ . The scalar product in  $\mathcal{H} \otimes \mathcal{H}$  corresponds to the Hilbert–Schmidt scalar product between operators

$$\langle\langle A|B\rangle\rangle = \text{Tr}[A^\dagger B], \quad (2)$$

and analogously the norm of vectors corresponds to the Frobenius norm. The following identities will be handy for calculations

$$A \otimes B|C\rangle\rangle = |ACB^T\rangle\rangle, \quad (3)$$

$$\text{Tr}_1[|A\rangle\rangle\langle\langle B|] = A^T B^*, \quad (4)$$

$$\text{Tr}_2[|A\rangle\rangle\langle\langle B|] = AB^\dagger, \quad (5)$$

where  $A^T$  and  $A^*$  denote the transposed operator and the complex conjugated operator of  $A$ , respectively, with respect to the basis used in Eq. (1).

From Eqs. (4) and (5) it is clear that for finite dimension  $d = \dim(\mathcal{H})$  the maximally entangled states are those corresponding to unitary operators scaled by  $1/\sqrt{d}$ , since these are the only pure states having maximally chaotic local states. For infinite dimensions we will consider Dirac-normalizable maximally entangled vectors.

## 3. Systems with finite-dimensional Hilbert space

Let us consider the set of  $d^2$  maximally entangled states  $d^{-1/2}|U(m,n)\rangle\rangle$ , corresponding to the *shift-and-multiply* unitary operators

$$U(m,n) = Z^m W^n, \quad (6)$$

where

$$Z = \sum_{j=0}^{d-1} e^{\frac{2\pi i}{d}j} |j\rangle\langle j|, \quad W = \sum_{j=0}^{d-1} |j \oplus 1\rangle\langle j|, \quad (7)$$

where  $\oplus$  denotes sum modulo  $d$ . The operators  $U(m,n)$  provide a projective irreducible representation of the Abelian group  $\mathbb{Z}_d \times \mathbb{Z}_d$ . It is easy to check [6] that the operators  $U(m,n)$  are an orthonormal basis for  $\mathcal{H} \otimes \mathcal{H}$ , whence the vectors  $d^{-1/2}|U(m,n)\rangle\rangle$  are a Bell basis, i.e., a maximally entangled orthonormal basis. This is precisely the basis used for quantum teleportation in Refs. [2,5]. For qubits ( $d=2$ ) it is the basis of Pauli matrices  $\{\frac{1}{\sqrt{2}}|I\rangle\rangle, \frac{1}{\sqrt{2}}|\sigma_x\rangle\rangle, \frac{1}{\sqrt{2}}|\sigma_y\rangle\rangle, \frac{1}{\sqrt{2}}|\sigma_z\rangle\rangle\}$ .

Our aim is now to find a unitary operator  $V$  on  $\mathcal{H} \otimes \mathcal{H}$  evolving a local basis into the Bell basis, more precisely such that

$$V|e_m\rangle\rangle = \frac{1}{\sqrt{d}}|U(m,n)\rangle\rangle, \quad (8)$$

where as a local basis we choose  $|e_m\rangle\rangle \otimes |n\rangle$ , the vector  $|e_j\rangle\rangle$  denoting the Fourier transformed vector of  $|j\rangle$ , with

$$|e_j\rangle\rangle \equiv \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{\frac{2\pi i}{d}nj} |n\rangle = F|j\rangle,$$

$$F \equiv \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{\frac{2\pi i}{d}nj} |n\rangle\langle j|. \quad (9)$$

In the following we will often use the short notation  $|\phi, \psi\rangle\rangle \doteq |\phi\rangle \otimes |\psi\rangle$  for tensor products of vectors.

A formal expression for  $V$  is readily given since we know its action on a complete orthonormal set

$$\begin{aligned} V &= \frac{1}{\sqrt{d}} \sum_{m,n=0}^{d-1} |U(m,n)\rangle\rangle \langle\langle e_m, n| \\ &= \frac{1}{\sqrt{d}} \sum_{m,n=0}^{d-1} \sum_{i,j=0}^{d-1} U(m,n)_{ij} |i, j\rangle\rangle \langle\langle e_m, n|, \end{aligned} \quad (10)$$

where, using Eqs. (6) and (7) we recover the matrix elements  $U(m,n)_{ij}$  as follows

$$U(m,n)_{ij} \doteq \langle\langle i, j|U(m,n)\rangle\rangle = e^{\frac{2\pi i}{d}im} \delta_{i\oplus n, j}. \quad (11)$$

This corresponds to the following expression for  $V$

$$\begin{aligned} V &= \frac{1}{\sqrt{d}} \sum_{m,n,i=0}^{d-1} e^{\frac{2\pi i}{d}im} |i\rangle\langle e_m| \otimes |n \oplus i\rangle\langle n| \\ &= \frac{1}{\sqrt{d}} \sum_{n,i=0}^{d-1} |i\rangle\langle i| \otimes |n \oplus i\rangle\langle n|. \end{aligned} \quad (12)$$

Thus we have

$$V = \sum_{i=0}^{d-1} |i\rangle\langle i| \otimes W^i, \quad (13)$$

namely the unitary  $V$  is a *controlled-unitary* transformation, corresponding to choosing (coherently) among the unitary transformations  $W^i$  via state preparation of the first system in  $\mathcal{H} \otimes \mathcal{H}$ . The transformation in Eq. (13) generalizes to dimension  $d > 2$  the well-known controlled-NOT gate for qubits. The present result is also in agreement with a similar one implicit in Ref. [7].

In the next section we will see how the present controlled-unitary evolution can be generalized to infinite dimensions, and how the local-Bell connection is achieved by a very common device: the beam splitter.

#### 4. Continuous variables

The derivation of Section 3 cannot be generalized straightforwardly, since the group  $\mathbb{Z}_d \times \mathbb{Z}_d$  has no extension to  $d = \infty$ . However, we will see how the same construction can be carried out for “continuous variables”—i.e., for continuous spectrum and bosonic modes—corresponding to an actual quantum optical implementation.

Let us consider two bosonic modes, with creation and annihilation operators denoted by  $a^\dagger, b^\dagger$ , and  $a, b$ , respectively. In the Hilbert space of each mode consider the complete sets of Dirac-normalized eigenvectors  $\{|x\rangle_0\}$  and  $\{|x\rangle_{\frac{\pi}{2}}\}$  of the quadratures  $X_0 = \frac{1}{2}(a^\dagger + a)$  and  $X_{\frac{\pi}{2}} = \frac{i}{2}(b^\dagger - b)$ . Consider the following unitary operator

$$C = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy |\Delta(x, y)\rangle\rangle_0 \langle x| \otimes \langle \frac{\pi}{2}| y|, \quad (14)$$

where  $\Delta(x, y) \equiv e^{-2ixX_{\frac{\pi}{2}}} e^{2iyX_0} = e^{-ixy} D(x + iy)$ ,  $D(\alpha) \doteq e^{\alpha a^\dagger - \alpha^* a}$  denoting the displacement operator. Notice that in the present infinite-dimensional setting the notation  $|A\rangle\rangle$  in Eq. (1) corresponds to  $|A\rangle \doteq A \otimes I |I\rangle\rangle$  with  $|I\rangle\rangle$  denoting the generalized vector  $|I\rangle\rangle = \sum_{n=0}^{\infty} |n\rangle \otimes |n\rangle \equiv \int_{\mathbb{R}} dx |x\rangle_0 \otimes |x\rangle_0$ , where in the representation of eigenstates of  $a^\dagger a$  the transposition of mode operators is given by  $a^T = a^\dagger$ .

The operator  $C$  can be considered as the infinite-dimensional analog of the operator  $V$  in Eq. (10), with the orthonormal basis  $\{|m\rangle\}$  replaced by the continuous Dirac set  $\{|x\rangle_0\}$ . The basis  $\frac{1}{\sqrt{\pi}} |\Delta(x, y)\rangle\rangle$  is indeed maximally entangled, with orthogonality relations  $\langle\langle \Delta(x, y) | \Delta(x', y') \rangle\rangle = \pi \delta(x - x') \delta(y - y')$ . The following calculation proves that  $C$  is a controlled-unitary transformation

$$\begin{aligned} C &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dt \\ &\quad \times e^{-2ixX_{\frac{\pi}{2}}} \otimes e^{2iyX_0} |t\rangle_{00} \langle x| \otimes |t\rangle_{0\frac{\pi}{2}} \langle y| \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dt \\ &\quad \times (I \otimes e^{2iyt}) |t+x\rangle_{00} \langle x| \otimes |t\rangle_{0\frac{\pi}{2}} \langle y| \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dy \int_{\mathbb{R}} dt e^{-2itX_{\frac{\pi}{2}}} \otimes |t\rangle_{0\frac{\pi}{2}} \langle y| e^{2iyt} \\ &= \left( \int_{\mathbb{R}} dt e^{-2itX_{\frac{\pi}{2}}} \otimes |t\rangle_{00} \langle t| \right) \\ &\quad \times \left( I \otimes \int_{\mathbb{R}} dy |y\rangle_{\frac{\pi}{2}\frac{\pi}{2}} \langle y| \right) \\ &= \int_{\mathbb{R}} dt e^{-2itX_{\frac{\pi}{2}}} \otimes |t\rangle_{00} \langle t|. \end{aligned} \quad (15)$$

Therefore, also for continuous variables we have a unitary operator that maximally entangles a factorized basis, and which corresponds to a controlled-unitary transformation.

#### 4.1. Achieving the controlled-unitary $C$

In this section we will give an optical scheme for achieving  $C$ , involving the use of parametric down-conversion and beam splitters.

For a suitable choice of phases of the input and output modes, a 50–50 beam-splitter can be represented by the unitary operator

$$V = e^{\frac{\pi}{4}(a^\dagger b - ab^\dagger)}. \quad (16)$$

The above unitary operator brings a local basis of two homodyne detectors into the Bell basis of a heterodyne detector [8]. In fact, consider the joint homodyne detection of two “orthogonal” quadratures, described by the generalized eigenvectors  $|x\rangle_0 \otimes |y\rangle_{\frac{\pi}{2}}$  of the quadratures  $X_0 \otimes X_{\frac{\pi}{2}}$ . It is immediate to show that  $|D(z)\rangle$  is the generalized eigenvector of the heterodyne photocurrent  $Z = a - b^\dagger$  corresponding to the complex eigenvalue  $z$ . We will now prove the following relation

$$V \left| \frac{x}{\sqrt{2}} \right\rangle_0 \otimes \left| \frac{y}{\sqrt{2}} \right\rangle_{\frac{\pi}{2}} = \left( \frac{2}{\pi} \right)^{1/2} |D(x + iy)\rangle. \quad (17)$$

First remind that the generalized eigenvectors  $|x\rangle_\phi$  of the generic quadrature  $X_\phi = \frac{1}{2}(e^{i\phi}a^\dagger + e^{-i\phi}a)$  can be written as

$$\begin{aligned} |x\rangle_\phi &= e^{-i\phi a^\dagger a} D(x)|0\rangle_0 \\ &= \left( \frac{2}{\pi} \right)^{1/4} e^{-i\phi a^\dagger a} D(x) e^{-a^{\dagger 2}/2} |0\rangle, \end{aligned} \quad (18)$$

where  $|0\rangle$  denotes the vacuum for mode  $a$ . Upon rewriting  $|x/\sqrt{2}\rangle_0 \otimes |y/\sqrt{2}\rangle_{\frac{\pi}{2}}$  via identity (18) in terms of operators acting on the vacuum, we apply the beam splitter operator  $V$  on the left, and after some algebra we obtain

$$\begin{aligned} V \left| \frac{x}{\sqrt{2}} \right\rangle_0 \otimes \left| \frac{y}{\sqrt{2}} \right\rangle_{\frac{\pi}{2}} &= \left( \frac{2}{\pi} \right)^{1/2} e^{-\frac{1}{2}(x^2+y^2) + a^\dagger(x+iy) - b^\dagger(x-iy)} e^{a^\dagger b^\dagger} \\ &\quad \times |0\rangle \otimes |0\rangle. \end{aligned} \quad (19)$$

Notice that  $e^{a^\dagger b^\dagger} |0\rangle \otimes |0\rangle = |I\rangle$ . Eq. (17) then follows immediately by using Eq. (3). We are now ready to derive the optical scheme for the controlled-unitary  $C$ . According to Eq. (17), we can rewrite  $V$  in the form

$$\begin{aligned} V(I \otimes e^{-i\frac{\pi}{2}b^\dagger b}) \left| \frac{x}{\sqrt{2}} \right\rangle_0 \otimes \left| \frac{y}{\sqrt{2}} \right\rangle_0 \\ = \left( \frac{2}{\pi} \right)^{1/2} e^{ixy} |\Delta(x, y)\rangle. \end{aligned} \quad (20)$$

Introducing the local squeezing transformation

$$S(r)|x\rangle = \sqrt{r}|rx\rangle, \quad (21)$$

we can rewrite Eq. (20) as follows

$$\begin{aligned} V(I \otimes e^{-i\frac{\pi}{2}b^\dagger b}) [S(2^{-1/2}) \otimes S(2^{-1/2})] \\ \times e^{-iX_0 \otimes X_0} (I \otimes e^{i\frac{\pi}{2}b^\dagger b}) \\ = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy |\Delta(x, y)\rangle_0 \langle x| \otimes \frac{\pi}{2} \langle y|. \end{aligned} \quad (22)$$

A realization of the transformation (21) is given by the unitary single-mode squeezing operator

$$S(r) = e^{\frac{1}{2} \log r (a^{\dagger 2} - a^2)}. \quad (23)$$

In l.h.s. of Eq. (22) the only unitary operator that has no direct physical interpretation is the exponential  $e^{-iX_0 \otimes X_0}$ . We will now write it as a product of physical unitary transformations. First, let us write the exponential in terms of bosonic operators

$$\begin{aligned} e^{-iX_0 \otimes X_0} &= e^{-\frac{i}{4}(a^\dagger + a)(b^\dagger + b)} \\ &= e^{-\frac{i}{4}(a^\dagger b^\dagger + ab + ab^\dagger + a^\dagger b)}. \end{aligned} \quad (24)$$

The operators  $K_x = \frac{1}{2}(a^\dagger b^\dagger + ab)$  and  $K_y = \frac{i}{2}(ab^\dagger + a^\dagger b)$  along with their commutator

$$K_z = i[K_x, K_y] = \frac{1}{4}(a^{\dagger 2} - a^2 + b^{\dagger 2} - b^2) \quad (25)$$

are the generators of the Lie algebra  $su(1, 1)$ . In terms of the generators of the Lie algebra, the exponential  $e^{-iX_0 \otimes X_0}$  is simply given by

$$e^{-iX_0 \otimes X_0} = e^{-\frac{i}{2}(K_x - iK_y)} = e^{-\frac{i}{2}K_-}. \quad (26)$$

Using the Pauli matrix realization of the Lie algebra  $K_{x/y} = \frac{i}{2}\sigma_{x/y}$  and  $K_z = \frac{1}{2}\sigma_z$ , one can easily derive the identity

$$e^{-\frac{i}{2}K_-} = e^{i\alpha K_x} e^{\beta K_y} e^{\gamma K_z}, \quad (27)$$

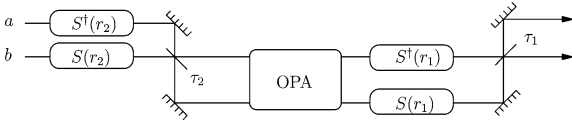


Fig. 1. Optical scheme to achieve the controlled unitary transformation (30) (Schrödinger picture).  $S(r)$  represents the squeezing transformation  $R(r) \doteq e^{\frac{1}{2} \log r (a^{\dagger 2} - a^2)}$ . The values of the squeezing parameters are  $r_1 = 2^{-1/2}$  and  $r_2 = (3/4)^{-1/4}$ . The beam splitters have transmissivities  $\tau_1 = \{4(2 - \sqrt{3})\}^{-1}$  and  $1/2$ . The optical parametric amplifier (OPA) has field-amplitude gain  $g = \{2(3 - 2\sqrt{3})\}^{-1}$ .

where

$$\begin{aligned} \alpha &= -2 \tanh^{-1}(2 - \sqrt{3}), \\ \beta &= -2 \tan^{-1}(2 - \sqrt{3}), \\ \gamma &= \log \frac{\sqrt{3}}{2}, \end{aligned} \quad (28)$$

namely one has

$$\begin{aligned} e^{-iX_0 \otimes X_0} &= e^{\frac{i}{2} \alpha (a^\dagger b^\dagger + ab)} e^{\frac{i}{2} \beta (a^\dagger b + ab^\dagger)} \\ &\quad \times e^{\frac{i}{4} \gamma (a^{\dagger 2} - a^2 + b^{\dagger 2} - b^2)}. \end{aligned} \quad (29)$$

Summarizing the above results, the controlled-unitary transformation  $C$  is realized as follows

$$\begin{aligned} C &= V(S(r_1) \otimes S(r_1)^\dagger) e^{-\frac{1}{2} \alpha (a^\dagger b^\dagger - ab)} e^{\frac{1}{2} \beta (a^\dagger b - ab^\dagger)} \\ &\quad \times (S(r_2)^\dagger \otimes S(r_2)), \end{aligned} \quad (30)$$

with  $r_1 = 2^{-1/2}$  and  $r_2 = (3/4)^{-1/4}$ . The overall scheme for the controlled unitary transformation is given in Fig. 1.

## 5. Conclusions

We have shown how a Bell measurement can be obtained from local measurements via an interaction of the controlled-unitary form. The considered Bell measurement is the one used in the original teleportation proposal of Ref. [2]. We conjecture that our

result holds more generally for every Bell measurement: however, a general proof would require a complete classification of all Bell measurements, which by itself is still an open problem. We have seen that for continuous variables a simple 50–50 beam-splitter achieves the factorization of the heterodyne Bell measurement, whereas a controlled-unitary interaction can be achieved by means of two squeezers, a parametric down-converter and two beam splitters. This result can be certainly of interest for applications to continuous-variable processing of quantum information.

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