CHAPTER 33

Homodyning as Universal Detection

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Abstract. Homodyne tomography—i.e., homodyning while scanning the local oscillator phase—is now a well assessed method for "measuring" the quantum state. In this paper I will show how it can be used as a kind of universal detection, for measuring generic field operators, however at expense of some additional noise. The general class of field operators that can be measured in this way is presented, and includes also operators that are inaccessible to heterodyne detection. The noise from tomographical homodyning is compared to that from heterodyning, for those operators that can be measured in both ways. It turns out that for some operators homodyning is better than heterodyning when the mean photon number is sufficiently small. Finally, the robustness of the method to additive phase-insensitive noise is analyzed. It is shown that just half photon of thermal noise would spoil the measurement completely.

1. Introduction

Homodyne tomography is the only viable method currently known for determining the detailed state of a quantum harmonic oscillator—a mode of the electromagnetic field. The state measurement is achieved by repeating many homodyne measurements at different phases φ with respect to the local oscillator (LO). The experimental work of the group in Eugene-Oregon [18] undoubtedly established the feasibility of the method, even though the earlier data analysis were based on a filtered procedure that affected the results with systematic errors. Later, the theoretical group in Pavia-Italy presented an exact reconstruction algorithm [7], which is the method currently adopted in actual experiments (see, for example, Refs. [16] and [17]). The reconstruction algorithm of Ref. [7] was later greatly simplified [5], so that it was possible also to recognize the feasibility of the method even for nonideal quantum efficiency $\eta < 1$ at the homodyne detector, and, at the same time, establishing lower bounds for η for any given matrix representation. After these first results, further theoretical progress has been made, understanding the mechanisms that underly the generation of statistical errors [2], thus limiting the sensitivity of the method. More recently, for $\eta = 1$ non trivial factorization formulas have been recognized [19, 14] for the "pattern functions" [15] that are necessary to reconstruct the photon statistics.

In this paper I will show how homodyne tomography can also be used as a method for measuring generic field operators. In fact, due to statistical errors, the measured matrix elements cannot be used to obtain expectations of field operators, and a different algorithm for analyzing homodyne data is needed suited to the particular field operator whose expectation one wants to estimate. Here, I will present an algorithm valid for any operator that admits a normal ordered expansion, giving the general class of operators that can be measured in this way, also as a function of the quantum efficiency η. Hence, from the same bunch of homodyne experimental data, now one can obtain not only the density matrix of the state, but also the expectation value of various field operators, including some operators that are inaccessible to heterodyne detection. However, the price to pay for such detection flexibility is that all measured quantities will be affected by noise. But, if one compares this noise with that from heterodyning (for those operators that can be measured in both ways), it turns out that for some operators homodyning is less noisy than heterodyning, at least for small mean photon numbers.

Finally, I will show that the method of homodyne tomography is quite robust to sources of additive noise. Focusing attention on the most common situation in which the noise is Gaussian and independent on the LO phase, I will show that this kind of noise produces the same effect of nonunit quantum efficiency at detectors. Generalizing the result of Ref. [5], I will give bounds for the overall rms noise level below which the tomographical reconstruction is still possible. I will show that the smearing effect of half photon of thermal noise in average is sufficient to completely spoil the measurement, making the experimental errors growing up unbounded.

2. Short Up-to-date Review on Homodyne Tomography

The homodyne tomography method is designed to obtain a general matrix element $\langle\psi|\hat{\varrho}|\varphi\rangle$ in form of expectation of a function of the homodyne outcomes at different phases with respect to the LO. In equations, one has

$$\langle\psi|\hat{\varrho}|\varphi\rangle = \int_0^{\pi}\frac{d\varphi}{\pi}\int_{-\infty}^{+\infty}dx\,p(x;\varphi)\,f_{\psi\varphi}(x;\varphi)\,, \tag{1}$$

where $p(x;\varphi)$ denotes the probability distribution of the outcome x of the quadrature $\hat{x}_\varphi = \frac{1}{2}\left(a^\dagger e^{i\varphi} + ae^{-i\varphi}\right)$ of the field mode with particle operators a and $a^\dagger$ at phase φ with respect to the LO. Notice that it is sufficient to average only over $\varphi \in [0,\pi]$, due to the symmetry $\hat{x}_{\varphi+\pi} = -\hat{x}_\varphi$. One wants the function $f_{\psi\varphi}(x;\varphi)$ bounded for all x, whence every moment will be bounded for any possible (*a priori* unknown) probability distribution $p(x;\varphi)$. Then, according to the central-limit theorem, one is guaranteed that the integral in Eq. (1) can be sampled statistically over a sufficiently large set of data, and the average values for different experiments will be Gaussian distributed, allowing estimation of confidence intervals. If, on the other hand, the kernel $f_{\psi\varphi}(x;\varphi)$ turns out to be unbounded, then we will say that the matrix element cannot be measured by homodyne tomography.

The easiest way to obtain the integral kernel $f_{\psi\varphi}(x;\varphi)$ is starting from the operator identity

$$\hat{\varrho} = \int\frac{d^2\alpha}{\pi}\,\mathrm{Tr}(\hat{\varrho}e^{-\overline{\alpha}a+\alpha a^\dagger})\,e^{-\alpha a^\dagger+\overline{\alpha}a} \tag{2}$$

which, by changing to polar variables $\alpha = (i/2)ke^{i\varphi}$, becomes

$$\hat{\varrho} = \int_0^{\pi}\frac{d\varphi}{\pi}\int_{-\infty}^{+\infty}\frac{dk\,|k|}{4}\,\mathrm{Tr}(\hat{\varrho}e^{ik\hat{x}_\varphi})\,e^{-ik\hat{x}_\varphi}\,. \tag{3}$$

Equation (2) is nothing but the operator form of the Fourier-transform relation between Wigner function and characteristic function: it can also be considered as an operator form of the Moyal identity

$$\int\frac{d^2z}{\pi}\langle k|\hat{D}^\dagger(z)|m\rangle\langle l|\hat{D}(z)|n\rangle = \langle k|n\rangle\langle l|m\rangle\,. \tag{4}$$

The trace-average in Eq. (3) can be evaluated in terms of $p(x,\varphi)$, using the complete set $\{|x\rangle_\varphi\}$ of eigenvectors of $\hat{x}_\varphi$, and exchanging the integrals over x and k. One obtains

$$\hat{\varrho} = \int_0^{\pi}\frac{d\varphi}{\pi}\int_{-\infty}^{+\infty}dx\,p(x;\varphi)K(x-\hat{x}_\varphi)\,, \tag{5}$$

where the integral kernel $K(x)$ is given by

$$K(x) = -\frac{1}{2}\mathrm{P}\frac{1}{x^2} \equiv -\lim_{\varepsilon\to 0^+}\frac{1}{2}\mathrm{Re}\frac{1}{(x+i\varepsilon)^2}\,, \tag{6}$$

P denoting the Cauchy principal value. Taking matrix elements of both sides of Eq. (5) between vectors ψ and φ, we obtain the sampling formula we were looking for, namely

$$\langle\psi|\hat{\varrho}|\varphi\rangle = \int_0^\pi \frac{d\varphi}{\pi} \int_{-\infty}^{+\infty} dx\, p(x;\varphi)\langle\psi|K(x-\hat{x}_\varphi)|\varphi\rangle\,. \tag{7}$$

Hence, the matrix element $\langle\psi|\hat{\varrho}|\varphi\rangle$ is obtained by averaging the function $f_{\psi\varphi}(x;\varphi) \equiv \langle\psi|K(x-\hat{x}_\varphi)|\varphi\rangle$ over homodyne data at different phases φ. As we will see soon, despite $K(x)$ is unbounded, for particular vectors ψ and φ in the Hilbert space the matrix element $\langle\psi|K(x-\hat{x}_\varphi)|\varphi\rangle$ is bounded, and thus the integral (7) can be sampled experimentally.

Before analyzing specific matrix representations, I recall how the sampling formula (7) can be generalized to the case of nonunit quantum efficiency. Low efficiency homodyne detection simply produces a probability $p_\eta(x;\varphi)$ that is a Gaussian convolution of the ideal probability $p(x;\varphi)$ for $\eta = 1$ (see, for example, Ref. [3]). In terms of the generating functions of the $\hat{x}_\varphi$-moments one has

$$\int_{-\infty}^{+\infty} dx\, p_\eta(x;\varphi)e^{ikx} = \exp\left(-\frac{1-\eta}{8\eta}k^2\right)\int_{-\infty}^{+\infty} dx\, p(x;\varphi)e^{ikx}\,. \tag{8}$$

Upon substituting Eq. (8) into Eq. (3), and by following the same lines that lead us to Eq. (5), one obtains the operator identity

$$\hat{\varrho} = \int_0^\pi \frac{d\varphi}{\pi} \int_{-\infty}^{+\infty} dx\, p_\eta(x;\varphi)K_\eta(x-\hat{x}_\varphi)\,, \tag{9}$$

where now the kernel reads

$$K_\eta(x) = \frac{1}{2}\mathrm{Re}\int_0^{+\infty} dk\, k\, \exp\left(\frac{1-\eta}{8\eta}k^2 + ikx\right)\,. \tag{10}$$

The desired sampling formula for $\langle\psi|\hat{\varrho}|\varphi\rangle$ is obtained again as in Eq. (7), by taking matrix elements of both sides of Eq. (10). Notice that now the kernel $K_\eta(x)$ is not even a tempered distribution: however, as we will see immediately, the matrix elements of $K_\eta(x-\hat{x}_\varphi)$ are bounded for some representations, depending on the value of η. The matrix elements $\langle\psi|K_\eta(x-\hat{a}_\varphi)|\varphi\rangle$ are bounded if the following inequality is satisfied for all phases $\varphi\in[0,\pi]$

$$\eta > \frac{1}{1+4\varepsilon^2(\varphi)}\,, \tag{11}$$

where $\varepsilon^2(\varphi)$ is the harmonic mean

$$\frac{2}{\varepsilon^2(\varphi)} = \frac{1}{\varepsilon^2_\psi(\varphi)} + \frac{1}{\varepsilon^2_\varphi(\varphi)}\,, \tag{12}$$

and $\varepsilon_\upsilon^2(\varphi)$ is the "resolution" of the vector $|\upsilon\rangle$ in the $\hat{x}_\varphi$-representation, namely:

$$|_\varphi\langle x|\upsilon\rangle|^2 \simeq \exp\left[-\frac{x^2}{2\varepsilon_\upsilon^2(\varphi)}\right] . \tag{13}$$

In Eq. (13) the symbol $\simeq$ stands for the leading term as a function of x, and $|x\rangle_\varphi \equiv e^{ia^\dagger a\varphi}|x\rangle$ denote eigen-ket of the quadrature $\hat{x}_\varphi$ for eigenvalue x. Upon maximizing Eq. (11) with respect to φ one obtains the bound

$$\eta > \frac{1}{1+4\varepsilon^2} , \qquad \varepsilon^2 = \min_{\varphi\in[0,\pi]} \{\varepsilon^2(\varphi)\} . \tag{14}$$

One can easily see that the bound is $\eta > 1/2$ for both number-state and coherent-state representations, whereas it is $\eta > (1+s^2)^{-1} \geq 1/2$ for squeezed-state representations with minimum squeezing factor $s < 1$. On the other hand, for the quadrature representation one has $\eta > 1$, which means that this matrix representation cannot be measured. The value $\eta = 1/2$ is actually an absolute bound for all representations satisfying the "Heisenberg relation" $\varepsilon(\varphi)\varepsilon(\varphi+\frac{\pi}{2}) \geq \frac{1}{4}$ with the equal sign, which include all known representations (for a discussion on the existence of exotic representations see Ref. [4]). Here, I want to emphasize that the existence of such a lower bound for quantum efficiency is actually of fundamental relevance, as it prevents measuring the wave function of a single system using schemes of weak repeated indirect measurements on the same system [10].

At the end of this section, from Ref. [5] I report for completeness the kernel $\langle n|K(x-\hat{x}_\varphi)|m\rangle$ for matrix elements between number eigenstates. One has

$$\langle n|K_\eta(x-\hat{x}_\varphi)|n+d\rangle = e^{-id\varphi}2\kappa^{d+2}\sqrt{\frac{n!}{(n+d)!}}e^{-\kappa^2x^2} \tag{15}$$
$$\times\sum_{\nu=0}^{n}\frac{(-)^\nu}{\nu!}\binom{n+d}{n-\nu}(2\nu+d+1)!\kappa^{2\nu}\mathrm{Re}\left\{(-i)^dD_{-(2\nu+d+2)}(-2i\kappa x)\right\} ,$$

where $\kappa = \sqrt{\eta/(2\eta-1)}$, and $D_\sigma(z)$ denotes the parabolic cylinder function. For $\eta = 1$ the kernel factorizes as follows [19, 14]

$$\langle n|K(x-\hat{x}_\varphi)|n+d\rangle$$
$$= e^{-id\varphi}[2xu_n(x)v_{n+d}(x) - \sqrt{n+1}u_{n+1}(x)v_{n+d}(x) - \sqrt{m+1}u_n(x)v_{n+d+1}(x)] , \tag{16}$$

where $u_n(x)$ and $v_n(x)$ are the regular and irregular energy eigen-functions of the harmonic oscillator

$$u_j(x) = \frac{1}{\sqrt{j!}} \left(x - \frac{\partial_x}{2} \right)^j \left(\frac{2}{\pi} \right)^{1/4} e^{-x^2},$$

$$v_j(x) = \frac{1}{\sqrt{j!}} \left(x - \frac{\partial_x}{2} \right)^j (2\pi)^{1/4} e^{-x^2} \int_0^{\sqrt{2}x} \mathrm{d}t \, e^{t^2}. \qquad (17)$$

3. Measuring Generic Field Operators

Homodyne tomography provides the maximum achievable information on the quantum state, and, in principle, the knowledge of the density matrix should allow one to calculate the expectation value $\langle \hat{O} \rangle = \mathrm{Tr}[\hat{O}\hat{\varrho}]$ of any observable $\hat{O}$. However, this is generally true only when one has an analytic knowledge of the density matrix, but it is not true when the matrix has been obtained experimentally. In fact, the Hilbert space is actually infinite dimensional, whereas experimentally one can achieve only a finite matrix, each element being affected by an experimental error. Notice that, even though the method allows one to extract *any* matrix element in the Hilbert space from the same bunch of experimental data, however, it is the way in which errors converge in the Hilbert space that determines the actual possibility of estimating the trace $\mathrm{Tr}[\hat{O}\hat{\varrho}]$. To make things more concrete, let us fix the case of the number representation, and suppose we want to estimate the average photon number $\langle a^\dagger a \rangle$. In Ref. [9] it has been shown that for nonunit quantum efficiency the statistical error for the diagonal matrix element $\langle n|\hat{\varrho}|n\rangle$ diverges faster than exponentially versus n, whereas for $\eta = 1$ the error saturates for large n to the universal value $\varepsilon_n = \sqrt{2/N}$ that depends only on the number N of experimental data, but is independent on both n and on the quantum state. Even for the unrealistic case $\eta = 1$, one can see immediately that the estimated expectation value $\langle a^\dagger a \rangle = \sum_{n=0}^{H-1} n\varrho_{nn}$ based on the measured matrix elements ϱ_{nn}, is not guaranteed to converge versus the truncated-space dimension H, because the error on ϱ_{nn} is nonvanishing versus n. Clearly in this way I am not proving that the expectation $\langle a^\dagger a \rangle$ is unobtainable from homodyne data, because matrix errors convergence depends on the chosen representation basis, whence the ineffectiveness of the method may rely in the data processing, more than in the actual information contained in the bunch of experimental data. Therefore, the question is: is it possible to estimate a generic expectation value $\langle \hat{O} \rangle$ directly from homodyne data, without using the measured density matrix? As we will see soon, the answer is positive in most cases of interest, and the procedure for estimating the expectation $\langle\hat{O}\rangle$ will be referred to as *homodyning the observable* $\hat{O}$.

By *homodyning the observable* $\hat{O}$ I mean averaging an appropriate kernel function $\mathcal{R}[\hat{O}](x;\varphi)$ (independent on the state $\hat{\varrho}$) over the experimental homodyne data, achieving in this way the expectation value of the observable $\langle\hat{O}\rangle$ for every state $\hat{\varrho}$. Hence, the kernel function $\mathcal{R}[\hat{O}](x;\varphi)$ is defined through the identity

$$\langle\hat{O}\rangle = \int_0^\pi \frac{d\varphi}{\pi}\int_{-\infty}^{+\infty} dx\, p(x;\varphi)\mathcal{R}[\hat{O}](x;\varphi)\,. \tag{18}$$

From the definition of $\mathcal{R}[\hat{O}](x;\varphi)$ in Eq. (18), and from Eqs. (2) and (3)—which generally hold true for any Hilbert-Schmidt operator in place of $\hat{\varrho}$—one obtains

$$\hat{O} = \int_0^\pi \frac{d\varphi}{\pi}\int_{-\infty}^{+\infty} dx\, \mathcal{R}[\hat{O}](x;\varphi)|x\rangle_{\varphi\varphi}\langle x|\ , \tag{19}$$

with the kernel $\mathcal{R}[\hat{O}](x;\varphi)$ given by

$$\mathcal{R}[\hat{O}](x;\varphi) = \mathrm{Tr}[\hat{O}K(x-\hat{x}_\varphi)]\ , \tag{20}$$

and $K(x)$ given in Eq. (6). The validity of Eq. (20), however, is limited only to the case of a Hilbert-Schmidt operator $\hat{O}$, otherwise it is ill defined. Nevertheless, one can obtain the explicit form of the kernel $\mathcal{R}[\hat{O}](x;\varphi)$ in a different way. Starting from the identity involving trilinear products of Hermite polynomials [11]

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2}\, H_k(x)\, H_m(x)\, H_n(x) = \frac{2^{\frac{m+n+k}{2}}\pi^{\frac{1}{2}}k!m!n!}{(s-k)!(s-m)!(s-n)!}\ , \quad \text{for } k+m+n=2s \text{ even}\ . \tag{21}$$

Richter proved the following nontrivial formula for the expectation value of the normally ordered field operators [20]

$$\langle a^{\dagger n}a^m\rangle = \int_0^\pi \frac{d\varphi}{\pi}\int_{-\infty}^{+\infty} dx\, p(x;\varphi)e^{i(m-n)\varphi}\frac{H_{n+m}(\sqrt{2}x)}{\sqrt{2^{n+m}\binom{n+m}{n}}}\ , \tag{22}$$

which corresponds to the kernel

$$\mathcal{R}[a^{\dagger n}a^m](x;\varphi) = e^{i(m-n)\varphi}\frac{H_{n+m}(\sqrt{2}x)}{\sqrt{2^{n+m}\binom{n+m}{n}}}\ . \tag{23}$$

This result can be easily extended to the case of nonunit quantum efficiency $\eta<1$, as the normally ordered expectation $\langle a^{\dagger n}a^m\rangle$ just gets an extra factor

$\eta^{\frac{1}{2}(n+m)}$. Therefore, one has

$$\mathcal{R}_\eta[a^{\dagger n}a^m](x;\varphi) = e^{i(m-n)\varphi}\frac{H_{n+m}(\sqrt{2}x)}{\sqrt{(2\eta)^{n+m}\binom{n+m}{n}}} , \tag{24}$$

where the kernel $\mathcal{R}_\eta[\hat{O}](x;\varphi)$ is defined as in Eq. (18), but with the experimental probability distribution $p_\eta(x;\varphi)$. From Eq. (24) by linearity on can obtain the kernel $\mathcal{R}_\eta[\hat{f}](x;\varphi)$ for any operator function $\hat{f}$ that has normal ordered expansion

$$\hat{f} \equiv f(a, a^\dagger) = \sum_{nm=0}^{\infty} f^{(n)}_{nm} a^{\dagger n} a^m . \tag{25}$$

From Eq. (24) one obtains

$$\begin{aligned}\mathcal{R}_\eta[\hat{f}](x;\varphi) &= \sum_{s=0}^{\infty}\frac{H_s(\sqrt{2}x)}{s!(2\eta)^{s/2}}\sum_{nm=0}^{\infty} f^{(n)}_{nm} e^{i(m-n)\varphi} n!m!\delta_{n+m,s} \\ &= \sum_{s=0}^{\infty}\frac{H_s(\sqrt{2}x)i^s}{s!(2\eta)^{s/2}}\frac{d^s}{dv^s}\bigg|_{v=0}\mathcal{F}[\hat{f}](v;\varphi),\end{aligned} \tag{26}$$

where

$$\mathcal{F}[\hat{f}](v;\varphi) = \sum_{nm=0}^{\infty} f^{(n)}_{nm}\binom{n+m}{m}^{-1}(-iv)^{n+m}e^{i(m-n)\varphi} . \tag{27}$$

Continuing from Eq. (26) one obtains

$$\mathcal{R}_\eta[\hat{f}](x;\varphi) = \exp\left(\frac{1}{2\eta}\frac{d^2}{dv^2} + \frac{2ix}{\sqrt{\eta}}\frac{d}{dv}\right)\bigg|_{v=0}\mathcal{F}[\hat{f}](v;\varphi) , \tag{28}$$

and finally

$$\mathcal{R}_\eta[\hat{f}](x;\varphi) = \int_{-\infty}^{+\infty}\frac{dw}{\sqrt{2\pi\eta^{-1}}}e^{-\frac{\eta}{2}w^2}\mathcal{F}[\hat{f}](w + 2ix/\sqrt{\eta};\varphi) . \tag{29}$$

Hence one concludes that the operator $\hat{f}$ can be measured by homodyne tomography if the function $\mathcal{F}[\hat{f}](v;\varphi)$ in Eq. (27) grows slower than $\exp(-\eta v^2/2)$ for $v \to \infty$, and the integral in Eq. (29) grows at most exponentially for $x \to \infty$ (assuming $p(x;\varphi)$ goes to zero faster than exponentially at $x \to \infty$).

In Table 3 I report the kernel $\mathcal{R}_\eta[\hat{O}](x;\varphi)$ for some operators $\hat{O}$. One can see that for the raising operator $\hat{e}_+$ the kernel diverges at $\eta = 1/2^+$, namely it can be measured only for $\eta > 1/2$. The operator $\hat{W}_s$ in the same table gives the generalized Wigner function $W_s(\alpha,\bar{\alpha})$ for ordering parameter s

through the identity $W_s(\alpha,\bar{\alpha}) = \mathrm{Tr}[\hat{D}(\alpha)\hat{\varrho}\hat{D}^\dagger(\alpha)\hat{W}_s]$. From the expression of $\mathcal{R}_\eta[\hat{W}_s](x;\varphi)$ it follows that by homodyning with quantum efficiency η one can measure the generalized Wigner function only for $s < 1-\eta^{-1}$: in particular, as already noticed in Refs. [5], the usual Wigner function for $s=0$ cannot be measured for any quantum efficiency [in fact one would have $\mathcal{R}_1[\hat{D}^\dagger(\alpha)\hat{W}_0\hat{D}(\alpha)](x;\varphi) = K[x - \mathrm{Re}(\alpha e^{-i\varphi})]$, with $K(x)$ unbounded as given in Eq. (6)].

	$\hat{O}$	$\mathcal{R}_\eta[\hat{O}](x;\varphi)$
(1)	$a^{\dagger n} a^m$	$e^{i(m-n)\varphi}\dfrac{H_{n+m}(\sqrt{2}x)}{\sqrt{2^{n+m}}\binom{n+m}{n}}$
(2)	a	$2e^{i\varphi}x$
(3)	a^2	$e^{2i\varphi}(4x^2-1)$
(4)	$a^\dagger a$	$2x^2 - \frac{1}{2}$
(5)	$(a^\dagger a)^2$	$\frac{8}{3}x^4 - 2x^2$
(6)	$:\hat{D}^\dagger(\alpha) :\doteq e^{-\alpha a^\dagger} e^{\overline{\alpha} a}$	$\frac{\exp[-\frac{1}{2\eta}(\overline{\alpha}e^{i\varphi})^2 + \frac{2x}{\sqrt{\eta}}\overline{\alpha}e^{i\varphi}]}{1+\frac{\alpha}{\overline{\alpha}}e^{-2i\varphi}} + \frac{\exp[-\frac{1}{2\eta}(\alpha e^{-i\varphi})^2 - \frac{2x}{\sqrt{\eta}}\alpha e^{-i\varphi}]}{1+\frac{\overline{\alpha}}{\alpha}e^{2i\varphi}}$
(7)	$\hat{e}_+ \doteq a^\dagger \frac{1}{\sqrt{1+a^\dagger a}}$	$2xe^{-i\varphi}\frac{1}{\sqrt{2\pi\eta}}\int_{-\infty}^{+\infty} dv \times \frac{e^{-v^2}}{(1+z)^2}\Phi\left(2,\frac{3}{2};\frac{x^2}{1+z^{-1}}\right)$, $z = \frac{e^{-v^2}-1}{2\eta}$
(8)	$\hat{W}_s \doteq \frac{2}{\pi(1-s)}\left(\frac{s+1}{s-1}\right)^{a^\dagger a}$	$\displaystyle\int_0^\infty dt \frac{2e^{-t}}{\pi(1-s)-\frac{1}{\eta}} \cos\left(2\sqrt{\frac{2t}{(1-s)-\frac{1}{\eta}}}x\right)$
(9)	$\lvert n+d\rangle\langle n\rvert$	$\langle n\lvert K(x-\hat{x}_\varphi)\rvert n+d\rangle$ in Eqs. (15) and (16)

Table 1: Kernel $\mathcal{R}_\eta[\hat{O}](x;\varphi)$, as defined in Eq. (18), for some operators $\hat{O}$. [The symbol $\Phi(a,b;x)$ denotes the customary confluent hypergeometric function.]

3.1. *Comparison between homodyne tomography and heterodyning*

We have seen that from the same bunch of homodyne tomography data, not only one can recover the density matrix of the field, but also one can measure any field observable $\hat{f} \equiv f(a,a^\dagger)$ having *normal ordered* expansion $\hat{f} \equiv f^{(n)}(a,a^\dagger) = \sum_{nm=0}^\infty f_{nm}^{(n)} a^{\dagger n} a^m$ and bounded integral in Eq. (29)—this holds true in particular for any polynomial function of the annihilation and creation operators. This situation can be compared with the case of heterodyne detection, where again one measures general field observables, but admitting *anti-normal ordered* expansion $\hat{f} \equiv f^{(a)}(a, a^\dagger) = \sum_{nm=0}^{\infty} f_{nm}^{(a)} a^m a^{\dagger n}$, in which case the expectation value is obtained through the heterodyne average

$$\langle \hat{f} \rangle = \int \frac{d^2\alpha}{\pi} f^{(a)}(\alpha, \overline{\alpha}) \langle \alpha | \hat{\varrho} | \alpha \rangle \ . \tag{30}$$

For $\eta = 1$ the heterodyne probability is just the Q-function $Q(\alpha, \overline{\alpha}) = \frac{1}{\pi}\langle \alpha | \hat{\varrho} | \alpha \rangle$, whereas for $\eta = 1$ it will be Gaussian convoluted. As shown by Baltin [1], generally the anti-normal expansion either is not defined, or is *not consistent* on the Fock basis, namely $f^{(a)}(a, a^\dagger)|n\rangle$ has infinite norm or is different from $\hat{f}(a, a^\dagger)|n\rangle$ for some $n \geq 0$. In particular, let us focus attention on functions of the number operator $f(a^\dagger a) = \sum_{l=0}^{\infty} c_l (a^\dagger a)^l$, $f^{(n)}(a^\dagger a) = \sum_{l=0}^{\infty} c_l^{(n)} a^{\dagger l} a^l$, $f^{(a)}(a^\dagger a) = \sum_{l=0}^{\infty} c_l^{(a)} a^l a^{\dagger l}$. Baltin has shown that [1]

$$\begin{aligned} c_l^{(n)} &= \frac{1}{l!} \int_{-\infty}^{+\infty} d\lambda\, g(\lambda)(e^{-i\lambda} - 1)^l = \sum_{k=0}^{l} \frac{(-)^{l-k} f(k)}{k!(l-k)!} \ , \\ c_l^{(a)} &= \frac{1}{l!} \int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda} g(\lambda)(1 - e^{i\lambda})^l = \sum_{k=0}^{l} \frac{(-)^k f(-k-1)}{k!(l-k)!} \ , \\ g(\lambda) &\doteq \int_{-\infty}^{+\infty} \frac{dx}{2\pi} f(x) e^{i\lambda x} \ . \end{aligned} \tag{31}$$

From Eqs. (31) one can see that the normal ordered expansion is always well defined, whereas the anti-normal ordering needs extending the domain of f to negative integers. However, even though the anti-normal expansion is defined, this does not mean that the expectation of $f(a^\dagger a)$ can be obtained through heterodyning, because the integral in Eq. (30) may not exist. Actually, this is the case when the anti-normal expansion is not consistent on the Fock basis. In fact, for the exponential function $f(a^\dagger a) = \exp(-\mu a^\dagger a)$ one has $f^{(a)}(|\alpha|^2) = e^\mu \exp[(1 - e^\mu)|\alpha|^2]$; on the Fock basis $f^{(a)}(a^\dagger a)|n\rangle$ is a binomial expansion with finite convergence radius, and this gives the consistency condition $|1 - e^\mu| < 1$. However, one can take the analytic continuation corresponding for $1 - e^\mu < 1$, which coincides with the condition that the integral in Eq. (30) exists for any state $\hat{\varrho}$ (the Q-function vanishes as $\exp(-|\alpha|^2)$ for $\alpha \to \infty$, at least for states with limited photon number). This argument can be extended by Fourier transform to more general functions $f(a^\dagger a)$, leading to the conclusion that there are field operators that cannot be heterodyne-measured, even though they have well defined anti-normal expansion, but the expansion is not consistent on the

Fock basis. As two examples, I consider the field operators $\hat{e}_+$ and $\hat{W}_s$ in Table 3. According to Eqs. (31) it follows that the operator $\hat{e}_+$ does not admit an anti-normal expansion, whence it cannot be heterodyne detected. This is in agreement with the fact that according to Table 3 we can homodyne $\hat{e}_+$ only for $\eta > 1/2$, and heterodyning is equivalent to homodyning with effective quantum efficiency $\eta = 1/2$ (which corresponds to the 3 dB noise due to the joint measurement [21]). The case of the operator $\hat{W}_s$ is different. It admits both normal-ordered and anti-normal-ordered forms: $\hat{W}_s = \frac{2}{\pi(1-s)} : \exp\left(-\frac{2}{1-s}a^\dagger a\right) := -\frac{2}{\pi(1+s)} : \exp\left(\frac{2}{1+s}a^\dagger a\right) :_A$, where $: \ldots :$ denotes normal ordering and $: \ldots :_A$ anti-normal. However, the consistency condition for anti-normal ordering is $2/(s+1) < 1$, with $s \le 1$, which implies that one can heterodyne $\hat{W}_s$ for $s > -1$, again in agreement with the value of s achievable by homodyne tomography at $\eta = 1/2$.

Now I briefly analyze the additional noise from homodyning field operators, and compare them with the heterodyne noise. For a complex random variable $z = u + iv$ the noise is given by the eigenvalues $N^{(\pm)} = \overline{|z|^2} - |\overline{z}|^2 \pm |\overline{z^2} - \overline{z}^2|$ of the covariance matrix. When homodyning the field, the random variable is $z \equiv 2e^{i\varphi}x$ [22] and the average over-line denotes the double integral over x and φ in Eq. (18). From Table (3) one has $\overline{z} = \langle a \rangle$, $\overline{z^2} = \langle a^2 \rangle$, $\overline{|z|^2} = 2\langle a^\dagger a \rangle + 1$, $\overline{e^{2i\varphi}} = 0$ [23]. In this way one finds that the noise from homodyning the field is $N^{(\pm)}_{hom}[a] = 1 + 2\langle a^\dagger a \rangle - |\langle a \rangle|^2 \pm |\langle a^2 \rangle - \langle a \rangle^2|$. On the other hand, when heterodyning, z becomes the heterodyne output photocurrent, whence $\overline{z} = \langle a \rangle$, $\overline{z^2} = \langle a^2 \rangle$, $\overline{|z|^2} = \langle a^\dagger a \rangle + 1$, and one has $N^{(\pm)}_{het}[a] = 1 + \langle a^\dagger a \rangle - |\langle a \rangle|^2 \pm |\langle a^2 \rangle - \langle a \rangle^2|$, so that the tomographical noise is larger than the heterodyne noise by a term equal to the average photon number, i. e.

$$N^{(\pm)}_{hom}[a] = N^{(\pm)}_{het}[a] + \langle a^\dagger a \rangle \ . \tag{32}$$

Therefore, homodyning the field is always more noisy than heterodyning it. On the other hand, for other field observables it may happen that homodyne tomography is less noisy than heterodyne detection. For example, one can easily evaluate the noise $N_{hom}[\hat{n}]$ when homodyning the photon number $\hat{n} = a^\dagger a$. The random variable corresponding to the photon number is $\nu(z) = \frac{1}{2}(|z|^2 - 1) \equiv 2x^2 - \frac{1}{2}$, and from Table 3 we see that the noise $N_{hom}[\hat{n}] \doteq \overline{\Delta\nu^2(z)}$ can be written as $N_{hom}[\hat{n}] = \langle \Delta\hat{n}^2 \rangle + \frac{1}{2}\langle \hat{n}^2 + \hat{n} + 1 \rangle$ [9]. When heterodyning the field, the random variable corresponding to the photon number is $\nu(z) = |z|^2 - 1$, and from the relation $\overline{|z|^4} = \langle a^{\dagger 2} a^2 \rangle$ one obtains $N_{het}[\hat{n}] \doteq \overline{\Delta\nu^2(z)} = \langle \Delta\hat{n}^2\rangle + \langle \hat{n}+1\rangle$, namely

$$N_{hom}[\hat{n}] = N_{het}[\hat{n}] + \frac{1}{2}\langle \hat{n}^2 - \hat{n} - 1\rangle \ . \tag{33}$$

We thus conclude that homodyning the photon number is less noisy than heterodyning it for sufficiently low mean photon number $\langle \hat{n}\rangle < \frac{1}{2}(1+\sqrt{5})$.

4. Homodyne Tomography in Presence of Additive Phase-insensitive Noise

In this section I consider the case of additive Gaussian noise, in the typical situation in which the noise is phase-insensitive. This kind of noise is described by a density matrix evolved by the master equation

$$\partial_t \hat{\varrho}(t) = 2\left[AL[a^\dagger] + BL[a]\right]\hat{\varrho}(t) \ , \tag{34}$$

where $L[\hat{c}]$ denotes the Lindblad super-operator $L[\hat{c}]\hat{\varrho} \doteq \hat{c}\hat{\varrho}\hat{c}^\dagger - \frac{1}{2}[\hat{c}^\dagger\hat{c}, \hat{\varrho}]_+$. Due to the phase invariance $L[ae^{-i\varphi}] = L[a]$ the dynamical evolution does not depend on the phase, and the noise is phase insensitive. From the evolution of the averaged field $\langle a\rangle_{out} \equiv \mathrm{Tr}[a\hat{\varrho}(t)] = g\langle a\rangle_{in} \equiv \mathrm{Tr}[a\hat{\varrho}(0)]$ with $g = \exp[(A-B)t]$, we can see that for $A > B$ Eq. (34) describes phase-insensitive amplification with field-gain g, whereas for $B > A$ it describes phase-insensitive attenuation, with $g < 1$. Concretely, for $A > B$ Eq. (34) models unsaturated parametric amplification with thermal idler [average photon number $\bar{m} = B/(A-B)$], or unsaturated laser action [A and B proportional to atomic populations on the upper and lower lasing levels respectively]. For $B > A$, on the other hand, the same equation describes a field mode damped toward the thermal distribution [inverse photon lifetime $\Gamma = 2(B-A)$, equilibrium photon number $\bar{m} = A/(B-A)$], or a loss $g < 1$ along an optical fiber or at a beam-splitter, or even due to frequency conversion [6]. The borderline case $A = B$ leaves the average field invariant, but introduces noise that changes the average photon number as $\langle a^\dagger a\rangle_{out} = \langle a^\dagger a\rangle_{in} + \bar{n}$, where $\bar{n} = 2At$. In this case the solution of Eq. (34) can be cast into the simple form

$$\hat{\varrho}(t) = \int \frac{d^2\beta}{\pi\bar{n}} \exp\left(-|\beta|^2/\bar{n}\right) \hat{D}(\beta)\hat{\varrho}(0)\hat{D}^\dagger(\beta) \ . \tag{35}$$

This is the *Gaussian displacement noise* studied in Refs. [13, 12] and commonly referred to as "thermal noise" [regarding the misuse of this terminology, see Ref. [12]], which can be used to model many kinds of undesired environmental effects, typically due to linear interactions with random classical fluctuating fields.

Eq. (34) has the following simple Fokker-Planck differential representation [8] in terms of the generalized Wigner function $W_s(\alpha,\bar{\alpha})$ for ordering parameter s

$$\partial_t W_s(\alpha,\bar{\alpha};t) = \left[Q(\partial_\alpha \alpha + \partial_{\bar{\alpha}}\bar{\alpha}) + 2D_s \partial^2_{\alpha,\bar{\alpha}}\right] W_s(\alpha,\bar{\alpha};t)\ , \tag{36}$$

where $Q = B - A$ and $2D_s = A + B + s(A - B)$. For nonunit quantum efficiency η and after a noise-diffusion time t the homodyne probability distribution $p_\eta(x;\varphi;t)$ can be evaluated as the marginal distribution of the Wigner function for ordering parameter $s = 1 - \eta^{-1}$, namely

$$p_\eta(x;\varphi;t) = \int_{-\infty}^{+\infty} dy W_{1-\eta^{-1}}\left((x+iy)e^{i\varphi}, (x-iy)e^{-i\varphi};t\right)\ . \tag{37}$$

The solution of Eq. (36) is the Gaussian convolution [8]

$$W_s(\alpha,\bar{\alpha};t) = \int \frac{d^2\beta}{\pi\delta_s^2} \exp\left[-\frac{|\alpha - g\beta|^2}{\delta_s^2}\right] W_s(\beta,\bar{\beta};0)\ ,$$
$$\delta_s^2 = \frac{D_s}{Q}(1 - e^{-2Qt})\ , \tag{38}$$

and using Eq. (37) one obtains the homodyne probability distribution

$$p_\eta(x;\varphi;t) = e^{Qt}\int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi\Delta^2_{1-\eta^{-1}}}} \exp\left[-\frac{(x' - g^{-1}x)^2}{2\Delta^2_{1-\eta^{-1}}}\right] p_\eta(x';\varphi), \tag{39}$$

where $\Delta^2_\eta = \frac{1}{2}g^{-2}\delta^2_{1-\eta^{-1}}$. It is easy to see that the generating function of the $\hat{x}_\varphi$-moments with the experimental probability $p_\eta(x;\varphi;t)$ can be written in term of the probability distribution $p(x;\varphi)$ for perfect homodyning as follows

$$\int_{-\infty}^{+\infty} dx\, p_\eta(x;\varphi;t)e^{ikx} =$$
$$\exp\left(-\frac{1}{2}g^2\Delta^2_\eta k^2 - \frac{1-\eta}{8\eta}g^2k^2\right)\int_{-\infty}^{+\infty} dx\, p(x;\varphi)e^{igkx}. \tag{40}$$

Eq. (40) has the same form of Eq. (8), but with the Fourier variable k multiplied by g and with an overall *effective quantum efficiency* η_* given by

$$\eta_*^{-1} = \eta^{-1} + 4\Delta^2_\eta = g^{-2}\eta^{-1} + \frac{2A}{B-A}(g^{-2} - 1)\ . \tag{41}$$

On the other hand, following the same lines that lead us to Eq. (9), we obtain the operator identity

$$\hat{\varrho} \equiv \hat{\varrho}(0) = \int_0^\pi \frac{d\varphi}{\pi}\int_{-\infty}^{+\infty} dx\, p_{\eta_*}(x;\varphi;t)K_{\eta_*}(g^{-1}x - \hat{x}_\varphi)\,, \tag{42}$$

which also means that when homodyning the operator $\hat{O}$ one should use $\mathcal{R}_{\eta_*}(g^{-1}x;\varphi)$ in place of $\mathcal{R}_\eta(x;\varphi)$, namely, more generally, one needs to re-scale the homodyne outcomes by the gain and use the effective quantum efficiency η_* in Eq. (41). In terms of the gain g and of the input-output photon numbers, the effective quantum efficiency reads

$$\eta_*^{-1} = \eta^{-1} + g^{-2}(2\langle a^\dagger a\rangle_{out} + \eta^{-1}) - (2\langle a^\dagger a\rangle_{in} + \eta^{-1}) \ . \tag{43}$$

In the case of pure displacement Gaussian noise ($A = B$), Eq. (43) becomes

$$\eta_*^{-1} = \eta^{-1} + 2\bar{n} \ , \tag{44}$$

which means that the bound $\eta_* > 1/2$ is surpassed already for $\bar{n} \geq 1$: in other worlds, it is just sufficient to have half photon of thermal noise to completely spoil the tomographic reconstruction.

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[22] Notice that for the complex random variable $z = 2e^{i\varphi}x$ the phase φ is a scanning parameter imposed by the detector. (Actually, the best way to experimentally scan the integral in Eq. (18) is just to pick up the phase φ at random.) Nevertheless, the argument of the complex number z is still a genuine random variable, because the sign of x is random, and depends on the value of φ. One has $\arg(z) = \varphi + \pi(1 - \mathrm{sgn}(x))$. For example, for any highly excited coherent state $|\alpha\rangle$ the probability distribution of $\arg(z)$ will approach a uniform distribution on $[\arg(\alpha) - \pi/2, \arg(\alpha) + \pi/2]$.

[23] One should remember that, the phase φ is imposed by the detector, and is uniformly scanned (randomly or not) in the interval $[0, \pi]$. This leads to $\overline{e^{2i\varphi}} = 0$, independently on the state $\hat{\varrho}$.