How Continuous Quantum Measurements in Finite Dimensions Are Actually Discrete

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We show that in finite dimensions a quantum measurement with a continuous set of outcomes can be always realized as a continuous random choice of measurements with a finite number of outcomes.

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When we measure the spin component along a magnetic field with a Stern-Gerlach apparatus, for spin-$1/2$ particles we have only two possible outcomes: spin up and spin down. This measurement is perfectly repeatable and can perfectly discriminate between the two orthogonal states $|\uparrow\rangle$ and $|\downarrow\rangle$. It is possible, however, to design an experiment with more than two outcomes, which discriminates optimally—though not perfectly—among three or more nonorthogonal states. Indeed, a four-outcome measurement on a two-level system is needed in the eavesdropping of a Bennett-Brassard 1984 protocol for cryptographic communication [1] or in the lab to perform an *informationally complete measurement* [2], which determines the quantum state from the measurement statistics.

What about performing a measurement with a *continuous* set of outcomes? This is the case of a measurement designed to optimally determine the "direction" of a spin [3], similarly to what we do in classical mechanics. Such a measurement produces a probability $p(\boldsymbol{n})d\boldsymbol{n}$ of the spin direction falling within the solid angle $d\boldsymbol{n}$ around the direction $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Indeed, the measurement of direction must be feasible [4]—though, in principle, inaccurate—otherwise, quantum mechanics would fail in describing what we normally observe in the macroscopic world. Actually, this is not the only interesting example of continuous-outcome measurement on a finite-level system: In fact, measurements of this kind have an endless number of applications, e.g., optimal state estimation [5], optimal alignment of directions [6] and reference frames [7], optimal phase estimation [8], and optimal design of atomic clocks [9].

In this Letter, we establish a fundamental property of quantum measurements with a continuous set of outcomes, namely, that for finite-level systems any such measurement is equivalent to a continuous random choice of measurements with a *finite* number of outcomes. This means that any physical quantity measured on a finite dimensional system is intrinsically discrete, while the continuum is pure classical randomness. For a spin-$1/2$ particle, this fact is well illustrated by the simple observation that the optimal measurement of direction can be realized by a customary Stern-Gerlach experiment where the magnetic field is randomly oriented. We emphasize that, in general, the discretization of physical quantities does not involve just von Neumann observables but, more generally, finite measurements with a number of outcomes larger than the Hilbert space dimension. Using the main result, we will show that any continuous measurement that optimizes some convex figure of merit (e.g., maximizing the mutual information or the Fisher information or, alternatively, minimizing a Bayes cost [8,10]) can be always replaced by a *single* measurement with finite outcomes, without affecting optimality.

Let us start by briefly reviewing the general theoretical description of measurements in quantum mechanics. Consider a quantum system [with Hilbert space $\mathcal{H}$ of dimension $\dim(\mathcal{H}) = d < \infty$], which undergoes a measurement with random outcome ω, distributed in the outcome space Ω. The probability distribution of the outcomes depends on the specific measuring apparatus via a *positive operator-valued measure* (POVM), namely, a function P that associates any subset $B \subseteq \Omega$ with a nonnegative operator $P(B)$, with normalization condition $P(\Omega) = I$ [11]. For a quantum system prepared in the state ρ, the probability of an outcome falling in the subset $B \subseteq \Omega$ is given by the Born rule $p(B) = \mathrm{Tr}[\rho P(B)]$. In the special case in which the measurement is finite, a random result i from a set of possible outcomes $\{i = 1, 2, \ldots, N\}$ is returned with probability $p_i = \mathrm{Tr}[\rho P_i]$, $P_i \geq 0$ being non-negative operators with normalization condition $\sum_{i=1}^N P_i = I$.

Before presenting the main result, in order to help intuition, we briefly analyze two simple prototypes of continuous-outcome measurement: the optimal measurement of the spin direction for a spin-$1/2$ particle and the optimal measurement of a phase shift.

The measurement of direction for a spin-$1/2$ particle has POVM P given by [3]

$$P(B) = \int_B \frac{d\boldsymbol{n}}{2\pi} |\boldsymbol{n}\rangle\langle\boldsymbol{n}|, \tag{1}$$

where $|\boldsymbol{n}\rangle$ is the eigenvector of $\boldsymbol{n} \cdot \boldsymbol{J}$ with eigenvalue

$+1/2$, $\boldsymbol{J}$ being the spin operator. It is simple to see that this measurement is equivalent to the randomization

$$P(B) = \int_{\mathbb{S}^2} \frac{d\boldsymbol{n}}{4\pi} E^{(\boldsymbol{n})}(B), \quad (2)$$

where $d\boldsymbol{n}/(4\pi)$ is the uniform probability distribution on the unit sphere $\mathbb{S}^2$, and $E^{(\boldsymbol{n})}$ is the POVM

$$E^{(\boldsymbol{n})}(B) = \chi_B(\boldsymbol{n})|\boldsymbol{n}\rangle\langle\boldsymbol{n}| + \chi_B(-\boldsymbol{n})|-\boldsymbol{n}\rangle\langle-\boldsymbol{n}|. \quad (3)$$

[$\chi_B(\boldsymbol{n})$ is the characteristic function of the set B: $\chi_B(\boldsymbol{n}) = 1$ for $\boldsymbol{n} \in B$; $\chi_B(\boldsymbol{n}) = 0$ otherwise.] The POVM $E^{(\boldsymbol{n})}$ represents a measurement of direction based on a Stern-Gerlach setup with the magnetic field oriented along $\boldsymbol{n}$: If the apparatus outputs "up," one assigns to the spin the direction $\boldsymbol{n}$; if "down," one assigns $-\boldsymbol{n}$. With this data processing, the probability of observing the spin within the region B is nonzero only if B contains at least one of the directions $\pm\boldsymbol{n}$. Hence, the continuous-outcome POVM (1) can be realized as a Stern-Gerlach measurement with random direction $\boldsymbol{n}$ of the magnetic field.

Another example of continuous-outcome measurement is that of phase estimation, where one wants to measure the phase shift $\phi \in [0, 2\pi)$ experienced by a quantum state under the action of the unitary evolution $U_\phi = \exp(iN\phi)$, with $N = \sum_{n=0}^{d-1} |n\rangle\langle n|$, $\{|n\rangle\}$ orthonormal basis for $\mathcal{H}$. The optimal POVM is given by [8]

$$P(B) = \int_B \frac{d\phi}{2\pi} |\phi\rangle\langle\phi|, \qquad |\phi\rangle = \sum_{n=0}^{d-1} e^{in\phi}|n\rangle \quad (4)$$

and is equivalent to the randomization $P(B) = \int_0^{2\pi} \frac{d\phi}{2\pi} E^{(\phi)}(B)$, where $E^{(\phi)}$ is the POVM $E^{(\phi)}(B) = \frac{1}{d} \times \sum_{n=0}^{d-1} \chi_B(\phi_n + \phi)|\phi_n + \phi\rangle\langle\phi_n + \phi|$, $\phi_n = \frac{2\pi n}{d}$.

We will now show that a continuous-outcome measurement in finite dimensions can always be realized in an analogous way, namely, as a continuous random choice of measurements with a finite number of outcomes. More precisely, we will prove the following

Theorem 1.—For any POVM P, the following decomposition holds:

$$P(B) = \int_{\mathcal{X}} dx p(x) E^{(x)}(B) \qquad \forall\ B \subseteq \Omega, \quad (5)$$

where $x \in \mathcal{X}$ is a suitable random variable, $p(x)$ a probability density, and, for every value of x, $E^{(x)}$ denotes a POVM with finite support, i.e., of the form

$$E(B) = \sum_{i=1}^{d^2} \chi_B(\omega_i) P_i, \quad (6)$$

$\{\omega_i \in \Omega\}$ being a set of points, and $\{P_i\}$ being a finite POVM with at most d^2 outcomes [12].

A POVM E as in Eq. (6) is nothing but the continuous data processing of the finite POVM $\{P_i\}$, with a function of the outcomes $f(i) = \omega_i$: If the apparatus outputs i, then one assigns to the measurement the outcome ω_i. The decomposition (5) shows that the continuous-outcome POVM P is achieved by randomly choosing a classical parameter $x \in \mathcal{X}$ and then performing the finitely supported POVM $E^{(x)}$, depending on x through the finite POVM $\{P_i^{(x)}\}$ and through the points $\{\omega_i^{(x)}\}$. Operationally, this corresponds to the following recipe: (i) Randomly draw a value of x according to $p(x)$; (ii) depending on x, measure the finite POVM $\{P_i^{(x)}\}$, thus getting the outcome i; (iii) for outcome i, assign to the continuous-outcome measurement the outcome $\omega_i^{(x)}$. As a first consequence, this simple recipe shows that, contrarily to a rather common belief (see, e.g., Ref. [13]), continuous-outcome quantum measurements in finite dimensions are as feasible as the finite ones.

The realization of the measurement of the "spin direction" given by Eq. (2) provides a concrete example of decomposition (5). In particular, the finitely supported POVM $E^{(\boldsymbol{n})}$ in Eq. (3) is illustrated in Fig. 1 for $\boldsymbol{n} = \mathbf{k}$. Notice that, in general, there may be different randomization schemes yielding the same continuous-outcome POVM: As an example, Fig. 1 illustrates another finitely supported POVM that allows one to reproduce the measurement of direction by simply randomizing the orientation of the Cartesian axes.

We now derive the main result. We fix both the quantum system and the outcome space Ω and consider the set $\mathcal{P}$ of all possible POVMs for these. This is a convex set, since,

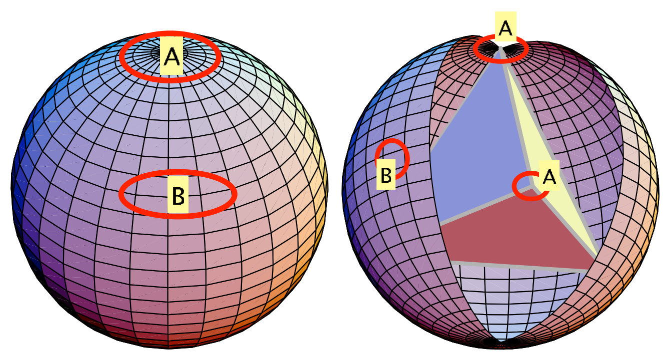

FIG. 1 (color online). Left: Illustration of the POVM $E^{(\boldsymbol{n})}$ in Eq. (3) as an example of finitely supported POVM $E^{(x)}$ in Eq. (6). In this specific example, the outcome space Ω is the unit sphere $\Omega \equiv \mathbb{S}^2$, the dimension of the Hilbert space is $d = 2$, and only two out of the four terms in Eq. (6) are nonvanishing: $P_1 = |\mathbf{k}\rangle\langle\mathbf{k}|$, $P_2 = |-\mathbf{k}\rangle\langle-\mathbf{k}|$, $P_3 = P_4 = 0$. The probability of finding the spin direction in a region $R \subseteq \Omega$ is zero for R missing the two poles, as B in the figure, and is possibly nonzero for R as A. Right: Another example of a finitely supported POVM for $d = 2$, corresponding to a symmetric informationally complete POVM [20]. The POVM is made of four elements P_i corresponding to ω_i at the vertices of a tetrahedron. The probability can be nonvanishing only if the region R contains at least one of these four points, such as in A, whereas it is always zero in situations as in B.

given any two POVMs P' and P'', their *convex combination* $P^{(\lambda)} = \lambda P' + (1-\lambda)P''$ for $\lambda \in [0,1]$ is still a POVM; namely, the whole segment joining P' and P'' is contained in $\mathcal{P}$. The *extremal points* of the convex set $\mathcal{P}$ are those POVMs that cannot be written as a convex combination of two different POVMs. Stated differently, a POVM $P \in \mathcal{P}$ is not extremal if and only if it is the midpoint of a segment completely contained in $\mathcal{P}$, i.e., if and only if there exist two distinct points $P', P'' \in \mathcal{P}$, $P' = P''$ such that $P = \frac{1}{2}(P' + P'')$. This is equivalent to the existence of a direction $Q \neq 0$ and a positive number $\epsilon > 0$ such that $P + tQ \in \mathcal{P}$ for any $t \in [-\epsilon, \epsilon]$. The standard name for the direction Q in convex analysis is *perturbation*. Here the perturbation Q is a function that associates to any subset $B \subseteq \Omega$ an operator $Q(B)$, fulfilling the three requirements: (i) $Q(B)$ is Hermitian for any subset $B \subseteq \Omega$; (ii) $Q(\Omega) = 0$; (iii) $P(B) + tQ(B) \geq 0$ for any $B \in \Omega$ and for any $t \in [-\epsilon, \epsilon]$.

If there exists a nonzero perturbation Q for P, then P is nonextremal: Using this criterion, we now establish that the extremal POVMs must necessarily have finite support; namely, they must be of the form of Eq. (6).

The proof takes advantage of the following.

Lemma 1.—Every POVM $P \in \mathcal{P}$ admits a density with unit trace; namely, for any POVM P, there exists a finite measure $\mu(d\omega)$ over Ω such that

$$P(B) = \int_B \mu(d\omega) M(\omega), \tag{7}$$

with $M(\omega) \geq 0$ and $\mathrm{Tr}[M(\omega)] = 1$ μ-almost everywhere.

Proof.—Consider the finite measure $\mu(d\omega)$ defined by $\mu(B) = \mathrm{Tr}[P(B)]$ $\forall$ $B \subseteq \Omega$. Since $P(B) \geq 0$, one has $P(B) \leq \mathrm{Tr}[P(B)]I = \mu(B)I$; namely, $P(B)$ is dominated by the measure $\mu(B)$. This implies that P admits a density $M(\omega)$ with respect to $\mu(d\omega)$ (see, e.g., p. 167 of Ref. [8]). Clearly, the density $M(\omega)$ has to be non-negative μ-almost everywhere. Moreover, for any $B \subseteq \Omega$, one has $\int_B \mu(d\omega) \equiv \mu(B) = \mathrm{Tr}[P(B)] = \int_B \mu(d\omega)\mathrm{Tr}[M(\omega)]$, whence $\mathrm{Tr}[M(\omega)] = 1$ μ-almost everywhere. ■

Thanks to this lemma, we can represent any POVM $P \in \mathcal{P}$ using its density. Now, to prove that an extremal POVM must be of the form (6), it is enough to show that for extremal POVMs the measure $\mu(d\omega)$ is concentrated on a finite set of outcomes $\{\omega_1, \ldots, \omega_{d^2}\}$, i.e., $\mu(B) = 0$ for any set $B \subseteq \Omega$ not containing any one of the points $\{\omega_i\}$. We recall the definition of *support* of a measure $\mu(d\omega)$ as the set of all points $\omega \in \Omega$ such that $\mu(B) > 0$ for any open set B containing ω.

Lemma 2.—Let $P \in \mathcal{P}$ be a POVM and $\mu(d\omega)$ the measure defined by $\mu(B) = \mathrm{Tr}[P(B)]$. If P is extremal, then the support of $\mu(d\omega)$ is finite and contains no more than d^2 points.

Proof.—Suppose that the support contains more than d^2 points. In this case, one can take $d^2 + 1$ points $\omega_i \in \Omega$ in the support and $d^2 + 1$ disjoint open sets $U_i \subset \Omega$, $i = 1, \ldots, d^2 + 1$, such that $\omega_i \in U_i$ for any i [14]. As a consequence, the space $L^\infty(\Omega, \mu)$ of integrable functions $f(\omega)$ that are bounded μ-almost everywhere has a dimension of at least $d^2 + 1$ [indeed, the characteristic functions $\chi_{U_i}(\omega)$ are a set of $d^2 + 1$ bounded and linearly independent functions]. Then consider the matrix elements $f_{ij}(\omega) = \langle i|M(\omega)|j\rangle$, where $M(\omega)$ is the POVM density of Eq. (7), and $|i\rangle, |j\rangle$ are elements of an orthonormal basis. Since the operators $M(\omega)$ are non-negative with unit trace almost everywhere, the functions $f_{ij}(\omega)$ are bounded almost everywhere, namely, $f_{ij} \in L^\infty(\Omega, \mu)$ $\forall$ i, j. Moreover, since the space $L^\infty(\Omega, \mu)$ has a dimension larger than d^2, it must contain at least one function $g(\omega) \neq 0$ that is linearly independent from the set $\{f_{ij}\}$. Using the Gram-Schmidt orthogonalization procedure, such a function g can always be chosen to be orthogonal to all f_{ij}, namely, $\int_\Omega \mu(d\omega) g^*(\omega) f_{ij}(\omega) = 0$ $\forall$ i, j. Finally, since $f^*_{ij}(\omega) = f_{ji}(\omega)$ $\forall$ i, j, such a g can be also chosen to be real. Now we claim that the Hermitian operators

$$Q(B) = \int_B \mu(d\omega) g(\omega) M(\omega) \tag{8}$$

provide a perturbation for the POVM P. Indeed, we have $Q(\Omega) = 0$ since $\langle i|Q(\Omega)|j\rangle = 0$ $\forall$ i, j:

$$\langle i|Q(\Omega)|j\rangle = \int_\Omega \mu(d\omega) g(\omega) \langle i|M(\omega)|j\rangle \tag{9}$$

$$= \int_\Omega \mu(d\omega) g(\omega) f_{ij}(\omega) = 0. \tag{10}$$

Moreover, since $g \in L^\infty(\Omega, \mu)$, there exists a positive number c such that $|g(\omega)| \leq c < \infty$ almost everywhere, thus implying that the operators $M(\omega)[1 + tg(\omega)]$ are almost everywhere non-negative for any $t \in [-\epsilon, \epsilon]$, with $\epsilon = 1/(2c)$. Hence, integrating over any subset B, we obtain that the operators $P(B) + tQ(B)$ are non-negative; namely, Q is a perturbation. Finally, Q is nonzero; otherwise, taking the trace of Eq. (8) and using that $\mathrm{Tr}[M(\omega)] = 1$ almost everywhere, we would get $0 = \mathrm{Tr}[Q(B)] = \int_B \mu(d\omega) g(\omega)$ $\forall$ B, thus implying $g = 0$, which is not possible by the definition of g. In conclusion, if the support of $\mu(d\omega)$ contains more than d^2 points, then the POVM P has a nonzero perturbation, whence it is not extremal. ■

Lemma 2 establishes that an extremal POVM has necessarily the form of Eq. (6); namely, it can be realized by measuring a finite POVM $\{P_i\}$ and conditionally declaring the measurement outcomes $\{\omega_i\}$. Using this fact, we readily obtain the proof of the main theorem.

Proof of Theorem 1.—Because of the Krein-Milman theorem of convex analysis, any point of a compact convex set is a continuous convex combination of points that are either extremal or limit of extremals. On the other hand, it is simple to prove that the POVMs form a compact set [15] and that any limit of extremal POVMs is a POVM of the form (6) [16,17]. ■

We now want to explore some consequences of decomposition (5) for optimization of POVMs and for quantum tomography. Optimizing a quantum measurement consists in finding the POVM P that maximizes the value of a figure of merit $\mathcal{F}[P]$—e.g., mutual or Fisher information, average fidelity, or any Bayes gain. In all of these cases, $\mathcal{F}[P]$ is convex, i.e., $\mathcal{F}[\lambda P' + (1-\lambda)P''] \le \lambda\mathcal{F}[P'] + (1-\lambda)\mathcal{F}[P'']$ for any $\lambda \in [0,1]$. Suppose now that a continuous-outcome POVM P is optimal for $\mathcal{F}$. Combining convexity of $\mathcal{F}$ with Eq. (5), one has

$$\mathcal{F}_{\max} = \mathcal{F}[P] \le \int_X dx p(x)\mathcal{F}[E^{(x)}] \le \mathcal{F}_{\max}, \qquad (11)$$

which implies $\mathcal{F}[E^{(x)}] = \mathcal{F}_{\max}$ for any x except at most a set of zero measure. This means that all of the finite POVMs $E^{(x)}$ are equally optimal: In particular, for any optimal continuous-outcome measurement, there is always an optimal measurement with a finite (no more than d^2) number of outcomes. In special situations, some explicit algorithms to find optimal finite measurements are known [13,18,19]. In particular, Ref. [18] shows that in many cases the *minimal* number of outcomes is larger than $d = \dim(\mathcal{H})$. Combined with the above result, this fact definitely proves that the quantum discretization cannot rely solely on von Neumann measurements.

Regarding quantum tomography, using the present analysis, one can make mathematically precise the common intuition that a continuous-outcome informationally complete measurement is equivalent to a tomography scan made of a random choice of observables—more generally, POVMs. Indeed, one can estimate the ensemble average of any operator A by using the two data processing $f_A(\omega)$ and $f_A(\omega_i^{(x)})$ for continuous-outcome POVM and tomography, respectively, as follows:

$$A = \int_\Omega d\omega f_A(\omega)M(\omega) = \int_X dx p(x) \sum_{i=1}^{d^2} f_A(\omega_i^{(x)})P_i^{(x)}.$$

In conclusion, in this Letter we showed that continuous-outcome quantum measurements can always be realized by performing finite measurements depending on a random classical parameter. Physical properties measured on finite-level quantum systems, such as spatial orientation of microscopic gyroscopes and time of atomic clocks, are then intrinsically discrete.

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[14] The existence of the disjoint open sets $U_i \ni \omega_i$ is guaranteed if the outcome space Ω is a Hausdorff space, which is always the case for the physical outcome spaces.
[15] An operational notion of distance between POVMs is given in terms of the difference between the pertaining expectations $\mathbb{E}_\rho(f) = \int_\Omega f(\omega)\mathrm{Tr}[\rho P(d\omega)]$ of any function $f(\omega)$. Then the Banach-Alaoglu theorem shows that $\mathcal{P}$ is compact with respect to such distance.
[16] For compact Ω, let $E^{(n)}(B) = \sum_{i=1}^{d^2} \chi_B(\omega_i^{(n)})P_i^{(n)}$ be a sequence of extremal POVMs with limit $E(B)$. Take a subsequence n_k such that the points $\{\omega_i^{(n_k)}\}$ converge to a point $\omega_i \in \Omega$ for any i and the POVMs $\{P_i^{(n_k)}\}$ converge to a POVM $\{P_i\}$. Hence, $E(B) = \sum_{i=1}^{d^2} \chi_B(\omega_i)P_i$; i.e., the limit is still of the form (6). For noncompact Ω, one can introduce an auxiliary compact space $\bar{\Omega} \supset \Omega$ and apply the same argument for the POVMs on it. Since $\mathcal{P}$ is a subset of the set of POVMs on $\bar{\Omega}$, we can decompose any $P \in \mathcal{P}$ as in Eq. (5), with $E^{(x)}$ possibly supported on $\bar{\Omega} - \Omega$. However, due to the normalization $P(\Omega) = I$, the POVMs $E^{(x)}$ with support outside Ω must have zero measure in the decomposition of P (see also [17]).
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