QUIT quantum information theory group

Quantum Mechanics as a "Syntactic Manual" for the Experiment

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On the Present Status of Quantum Mechanics

9 September 2005, Losinj, Hotel Aurora

On experimental science

In any experimental science we perform **experiments** to get information on the **state** of an **object system**.

Knowledge on such state will allow us to predict the results of forthcoming experiments on the same (similar) object system in a similar situation.

Since necessarily we work with only partial prior knowledge of both system and experimental apparatus, the rules for the experiment must be given in a probabilistic setting.

On what is an experiment

An experiment on a **object system** consists in making it interact with an **apparatus**.

The interaction between object and apparatus produces one of a **set of possible transformations** of the object, each one occurring with some probability.

Information on the **state** of the object system at the beginning of the experiment is gained from the knowledge of which transformation occurred, which is the **outcome** that is signaled by the apparatus.

Actions and outcomes

Experiment or "action": the action on the object system due to an experiment is the set $\mathbb{A} \equiv \{\mathscr{A}_j\}$ of possible transformations $\mathscr{A}_j$ having overall unit probability, with the apparatus signaling the outcome j labeling which transformation actually occurred.

States

State: A state ω for a physical system is a rule which provides the probability for any possible transformation within an experiment, namely:

$$\omega : \boldsymbol{state}, \quad \omega(\mathscr{A}) : probability\ that\ the\ transformation\ \mathscr{A}\ occurs$$

No experiment: the identical transformations occurs with probability one

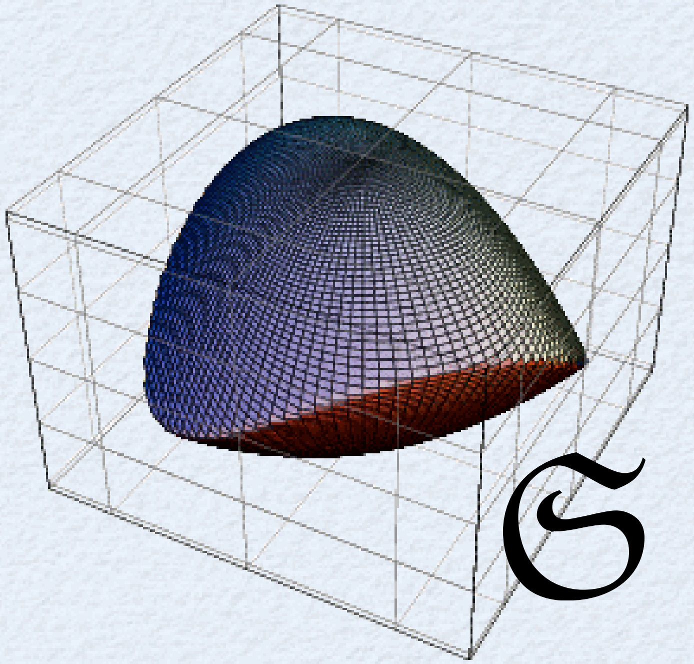
$$\omega(\mathscr{I}) = 1$$

Normalization:

$$\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$$

Convex structure of states

The possible states of a physical system make a convex set $\mathfrak{S}$, namely for any two states ω_1 and ω_2 we can consider the state ω which is the mixture of ω_1 with probability λ and of ω_2 with probability $1-\lambda$. We will write



$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2, \quad 0 \le \lambda \le 1,$$

for the state ω corresponding to the probability rule for transformations $\mathscr{A}$

$$\omega(\mathscr{A}) = \lambda\omega_1(\mathscr{A}) + (1-\lambda)\omega_2(\mathscr{A})$$

Affine dimension: $\mathrm{adm}(\mathfrak{S})$

Monoid of transformations

Transformations make a monoid: the composition $\mathscr{A} \circ \mathscr{B}$ of two transformations $\mathscr{A}$ and $\mathscr{B}$ is itself a transformation. Consistency of composition of transformations requires associativity, namely

$$\mathscr{C} \circ (\mathscr{B} \circ \mathscr{A}) = (\mathscr{C} \circ \mathscr{B}) \circ \mathscr{A}$$

There exists the identical transformation $\mathscr{I}$ which leaves the physical system invariant, and which for every transformation $\mathscr{A}$ satisfies the composition rule

$$\mathscr{I} \circ \mathscr{A} = \mathscr{A} \circ \mathscr{I} = \mathscr{A}$$

Independent systems and local transformations

Independent systems and local experiments: two physical systems are "independent" if on each system it is possible to perform "local experiments" for which on any joint state one has the commutativity of the pertaining transformations

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$$

$$(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) \doteq \mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} \circ \mathscr{C}^{(3)} \circ \ldots$$

Multipartite system: a collection of independent systems

Local state

For a multipartite system we define the local state $\omega|_n$ of the *n*-th system the state that gives the probability of any local transformation $\mathscr{A}$ on the *n*-th system with all other systems untouched, namely

$$\omega|_n(\mathscr{A}) \doteq \Omega(\mathscr{I}, \ldots, \mathscr{I}, \underbrace{\mathscr{A}}_{nth}, \mathscr{I}, \ldots)$$

Conditional state

When composing two transformations $\mathscr{A}$ and $\mathscr{B}$ the probability that $\mathscr{B}$ occurs conditioned that $\mathscr{A}$ happened before is given by the **Bayes rule**

$$p(\mathscr{B}|\mathscr{A}) = \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

Conditional state: the conditional state $\omega_{\mathscr{A}}$ gives the probability that a transformation $\mathscr{B}$ occurs on the physical system in the state ω after the transformation $\mathscr{A}$ occurred, namely

$$\omega_{\mathscr{A}}(\mathscr{B}) \doteq \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})} \qquad \omega_{\mathscr{A}} \doteq \frac{\omega(\cdot \circ \mathscr{A})}{\omega(\mathscr{A})}$$

Acausality

Notice that:

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)} \not\Longrightarrow \frac{\Omega(\cdot, \mathscr{B})}{\Omega(\mathscr{I}, \mathscr{B})} = \Omega(\cdot, \mathscr{I})$$

namely the occurrence of the transformation $\mathscr{B}$ on system 2 generally affects the conditional state on system 1, i. e.

$$\Omega_{\mathscr{I},\mathscr{B}}(\cdot, \mathscr{I}) \doteq \frac{\Omega(\cdot, \mathscr{B})}{\Omega(\mathscr{I}, \mathscr{B})} \neq \Omega(\cdot, \mathscr{I}) \equiv \omega|_1$$

Therefore, in order to guarantee acausality of local actions we need to require that any local action on a system is equivalent to the identity transformation on another independent system:

$$\forall \mathbb{A} \qquad \sum_{\mathscr{A}_j \in \mathbb{A}} \Omega(\cdot, \mathscr{A}_j) = \Omega(\cdot, \mathscr{I}) \equiv \omega|_1$$

Dynamical and informational equivalence

From the definition of conditional state we have:

- there are different transformations which produce the same state change, but generally occur with different probabilities
- there are different transformations which always occur with the same probability, but generally affect a different state change

Dynamical and informational equivalence

Dynamical equivalence of transformations: two transformations $\mathscr{A}$ and $\mathscr{B}$ are dynamically equivalent if

$$\omega_{\mathscr{A}} = \omega_{\mathscr{B}} \qquad \forall \omega \in \mathfrak{S}$$

Informational equivalence of transformations: two transformations $\mathscr{A}$ and $\mathscr{B}$ are informationally equivalent if

$$\omega(\mathscr{A}) = \omega(\mathscr{B}) \qquad \forall \omega \in \mathfrak{S}$$

Informational compatibility

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are informationally compatible (or coexistent) if for every state ω one has

$$\omega(\mathscr{A}) + \omega(\mathscr{B}) \leq 1$$

For any two coexistent transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ we define the transformation $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_2$ as the transformation corresponding to the event $e = \{1, 2\}$ namely the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred, but doesn't specify which one:

$$\forall \omega \in \mathfrak{S} \qquad \omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2)$$

$$\forall \omega \in \mathfrak{S} \qquad \omega_{\mathscr{A}_1 + \mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_2}$$

Informational compatibility

Multiplication by a scalar: for each transformation $\mathscr{A}$ the transformation $\lambda\mathscr{A}$ for $0 \leq \lambda \leq 1$ is defined as the transformation which is dynamically equivalent to $\mathscr{A}$ but occurs with probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$

Convex structure for transformations and actions

+ norm on transformation and approximability criterion

Banach algebra structure for transformations

Effect

We call ***effect*** an informational equivalence class $[\mathscr{A}]$ of transformations $\mathscr{A}$

duality

effects as positive linear l functionals over states:

$$l_{[\mathscr{A}]}(\omega) \doteq \omega(\mathscr{A})$$

Convex structure for effects

Observable

Observable: a set of effects $\mathbb{L} = \{l_i\}$ which is informationally equivalent to an action $\mathbb{A}$, namely such that there exists an action $\mathbb{A} = \{\mathscr{A}_j\}$ for which one has

$$l_i \in [\mathscr{A}_j] \; \forall j$$

Perfectly discriminable states $\{\omega_j\}$: there exists an observable $\mathbb{L} = \{l_i\}$ such that

$$l_i(\omega_j) = \delta_{ij}$$

Informational dimension $\mathrm{idm}(\mathfrak{S})$: maximal number of perfectly discriminable states

Informationally complete observable

Informationally complete observable: an observable $\mathbb{L} = \{l_i\}$ is informationally complete if any effect l can be written as linear combination of elements of $\mathbb{L}$, namely there exist coefficients $c_i(l)$ such that

$$l = \sum_{i=1}^{|\mathbb{L}|} c_i(l) l_i$$

affine dimension: $\mathrm{adm}(\mathfrak{S}) = |\mathbb{L}| - 1$, for $\mathbb{L}$ minimal informationally complete on $\mathfrak{S}$

Block representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}}) n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

Conditioning: fractional affine transformation

$$\boldsymbol{n}(\omega) \longrightarrow \boldsymbol{n}(\omega_{\mathscr{A}})$$

$$\boldsymbol{n}(\omega_{\mathscr{A}}) = \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{n}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})}$$

$$M_{ij}(\mathscr{A}) = \begin{pmatrix} q(\underline{\mathscr{A}}) & \boldsymbol{m}(\underline{\mathscr{A}}) \\ \boldsymbol{k}(\mathscr{A}) & \boldsymbol{M}(\mathscr{A}) \end{pmatrix}$$

Principle of local observability

For every composite system there exist informationally complete observables made only of local informationally complete observables.

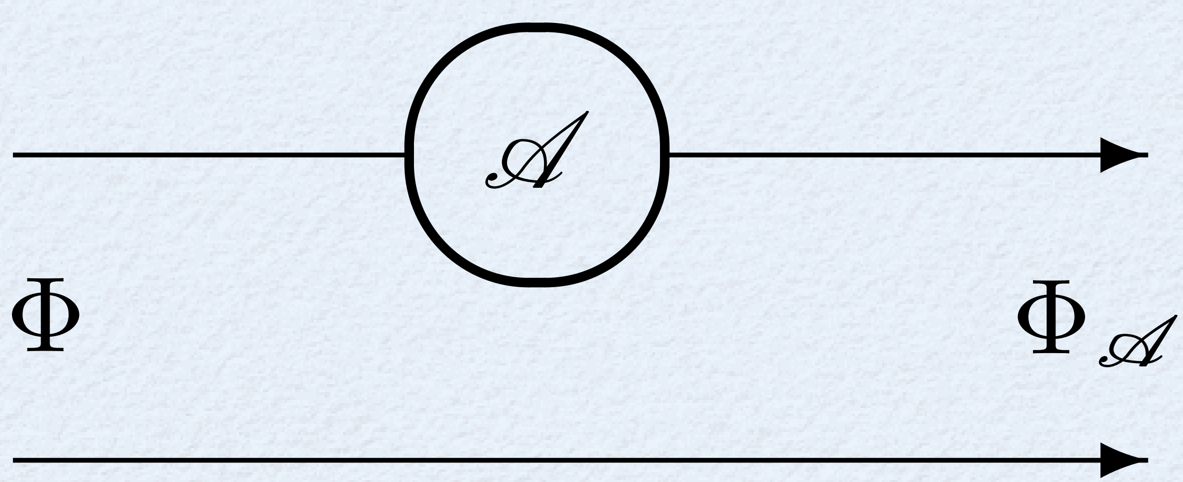
upper bound for the affine dimension of composite systems

$$\mathrm{adm}(\mathfrak{S}_{12}) \leq \mathrm{adm}(\mathfrak{S}_1)\,\mathrm{adm}(\mathfrak{S}_2) + \mathrm{adm}(\mathfrak{S}_1) + \mathrm{adm}(\mathfrak{S}_2)$$

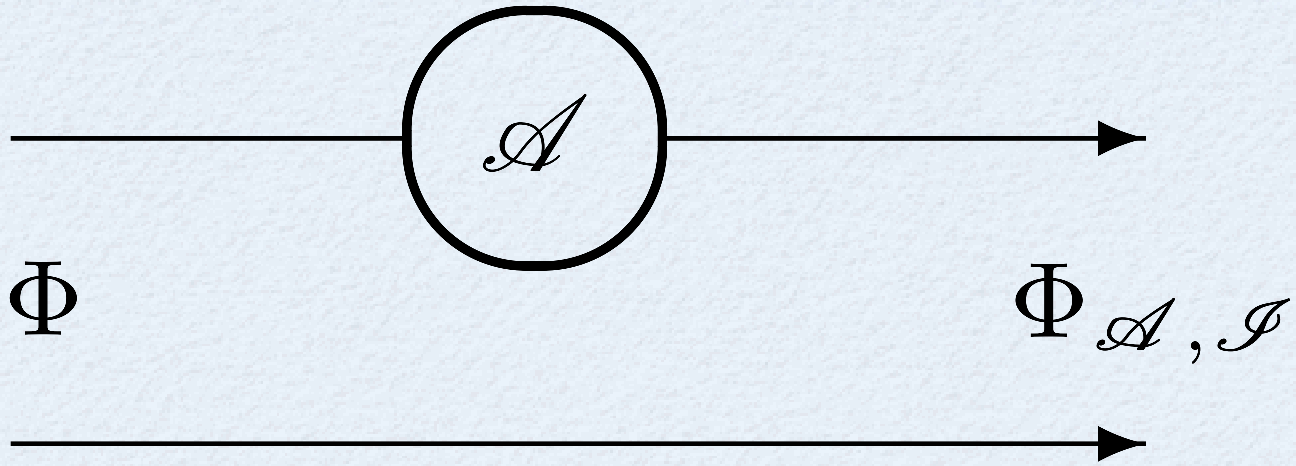
Faithful states

Dynamically faithful state: we say that a state Φ of a multipartite system is dynamically faithful for the *n*-th component system if when acting on it with a local transformation $\mathscr{A}$ the resulting conditioned state is in 1-to-1 correspondence with the dynamical equivalence class of $\mathscr{A}$, namely the following map is 1-to-1

$$\Phi_{\mathscr{I},\ldots,\mathscr{I},\mathscr{A},\mathscr{I},\ldots} \leftrightarrow [\mathscr{A}]_{dyn}$$



Existence of faithful states



$$\Phi_{\mathscr{A},\mathscr{I}} \leftrightarrow \mathscr{A}$$

lower bound for the affine dimension of a system composed of two identical systems

$$\mathrm{adm}(\mathfrak{S}^{\times 2}) \geq \mathrm{adm}(\mathfrak{S})[\mathrm{adm}(\mathfrak{S}) + 2]$$

First dimensionality identity: the tensor product

Local observability principle + faithful states

⇓

dimension of a system composed of two identical systems

$$\mathrm{adm}(\mathfrak{S}^{\times 2}) = \mathrm{adm}(\mathfrak{S})[\mathrm{adm}(\mathfrak{S}) + 2]$$

$$\dim(\mathsf{H} \otimes \mathsf{H})^2 - 1 = (\dim(\mathsf{H})^2 - 1)(\dim(\mathsf{H})^2 + 1)$$

$$\dim(\mathsf{H} \otimes \mathsf{H}) = \dim(\mathsf{H})^2$$

Second dimensionality identity: the Hilbert space

Realization of informationally complete observables from discriminating observables

For any bipartite system there exists a discriminating joint observable which is (minimal) informationally complete for one of the two components for almost all preparations of the other components.

$$\mathrm{adm}(\mathfrak{S}) + 1 \geq \mathrm{idm}(\mathfrak{S}^{\times 2})$$

$$\boxed{\mathrm{adm}(\mathfrak{S}) = \mathrm{idm}(\mathfrak{S})^2 - 1} \quad \textbf{!!}$$

Conjectured possible axioms

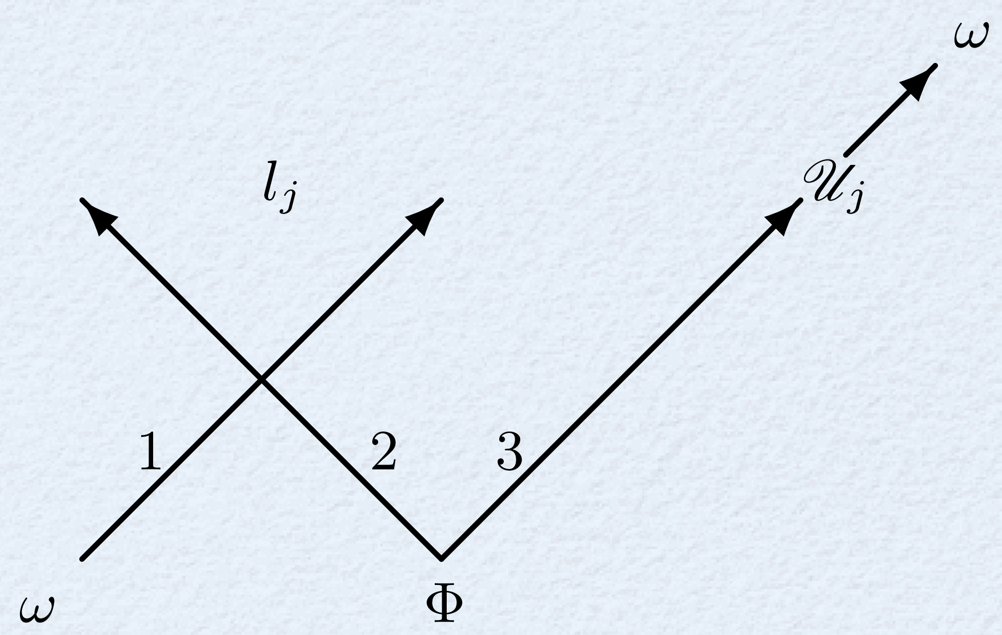
There exists *pure* faithful states

Conjectured possible axioms

Teleportation axiom

There exists a joint bipartite state Φ, a joint bipartite discriminating observable $\mathbb{L} = \{l_j\}$ and a set of deterministic indecomposable transformations $\{\mathscr{U}_j\}$ by which one can teleport all states ω as follows

$$\left.\frac{(\omega\Phi)(l_j, \cdot\mathscr{U}_j)}{(\omega\Phi)(l_j, \mathscr{I})}\right|_3 = \omega$$



Summary

experiment → state → informational equivalence

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independent systems → local state

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Bayes rule → conditional state → dynamical equivalence

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Local observability

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Faithful state

$\mathrm{adm}(\mathfrak{S}^{\times 2}) = \mathrm{adm}(\mathfrak{S})[\mathrm{adm}(\mathfrak{S}) + 2]$

\+ *Informationally complete from joint discriminating observable* → $\mathrm{adm}(\mathfrak{S}) = \mathrm{idm}(\mathfrak{S})^2 - 1$

effect

observable

discriminating observable

informationally complete observable

Bloch representation + affine transformations