Characterization and Engineering of Quantum Detectors and Processors

ICSSUR, Puebla, Mexico (June 9-14 2003)

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Quantum Optics & Information Group at Pavia

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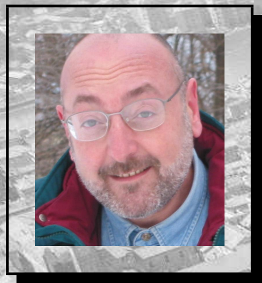

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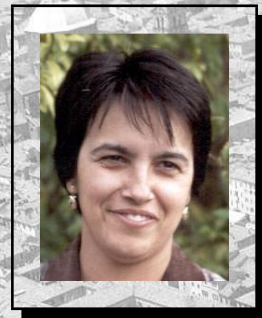

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M. Medici (secretary)

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  - Is there a special unitary U to be chosen for the ancilla-system interaction?

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- Covariant POVM

$$\mathrm{d}\,P_g = \mathrm{Tr}_2[\mathrm{d}\,B_g(I \otimes \sigma)] = \mathrm{d}\,g\,U_g \xi U_g^\dagger, \qquad \xi = V\sigma^\tau V^\dagger. \tag{2}$$

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Bell measurement from local measurements

- Bell measurement corresponding to the projective UIR of the Abelian group in d dimensions: $\mathbf{G} = \mathbf{Z}_d \times \mathbf{Z}_d$

$$U(m,n) = Z^m W^n, \quad Z = \sum_j \omega^j |j\rangle\langle j|, \quad W = \sum_k |k\rangle\langle k \oplus 1|, \quad \omega = e^{\frac{2\pi i}{d}}. \qquad (3)$$

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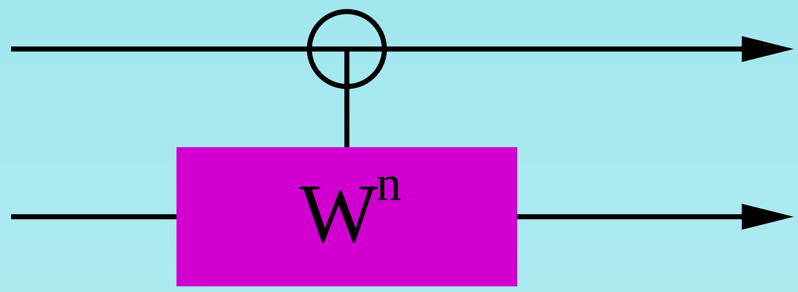
$$V = \sum_i |i\rangle\langle i| \otimes W^i. \qquad (5)$$



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- Nonorthogonal extremal POVM's are generally not connected by unitary transformations.

Convex structure of POVM's

Theorem 1 *The extremality of the POVM* $\mathbf{P} = (P_n)$ $n \in \mathsf{E} = \{1, 2, \ldots\}$ *is equivalent to the nonexistence of non trivial solutions* $\mathbf{D}$ *for the equation*

$$\sum_n D_n = 0, \quad \mathsf{Supp}(D_n), \mathsf{Rng}(D_n) \subseteq \mathsf{Supp}(P_n). \tag{6}$$

[G. M. D'Ariano and P. Lo Presti, (quant-ph/0301110)]

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Theorem 2 (Parthasaraty) *A POVM* $\mathbf{P}$ *is extremal iff the operators* $|v_i^{(n)}\rangle\langle v_j^{(n)}|$ *are linearly independent, for all eigenvectors* $|v_j^{(n)}\rangle$ *of* P_n.

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This means that a POVM with too many elements (i. e. $N > d^2$) will be decomposable into several POVM's, each with less than d^2 non-vanishing elements.

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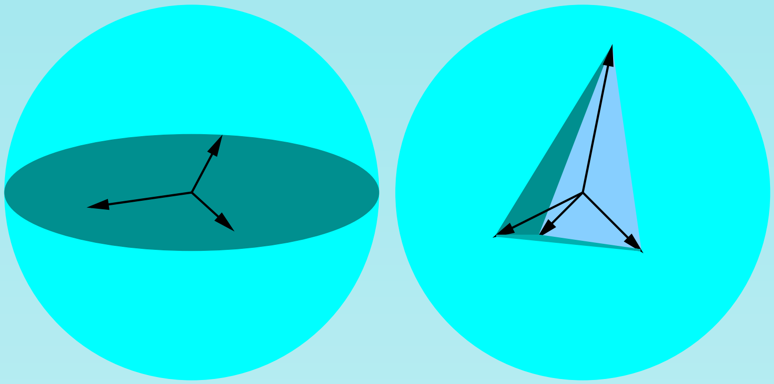

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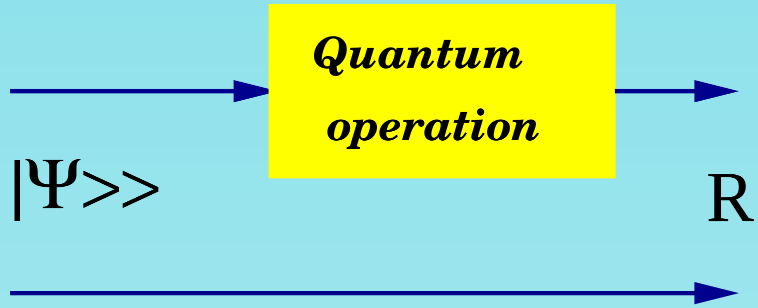
- For V of the controlled-U form $V = \sum_j |j\rangle\langle j| \otimes V_j$ it will be sufficient to find a covering for the manifold $SU(d)/U(1)^d$, such that

$$\min_j \|V_j - U\|_2 \le \epsilon/\sqrt{d}. \tag{13}$$

Howto characterize the operation of a device: entangled input

Quantum parallelism of entanglement **A single entangled input state $|\Psi\rangle\rangle$ is equivalent to scanning all states in parallel**.



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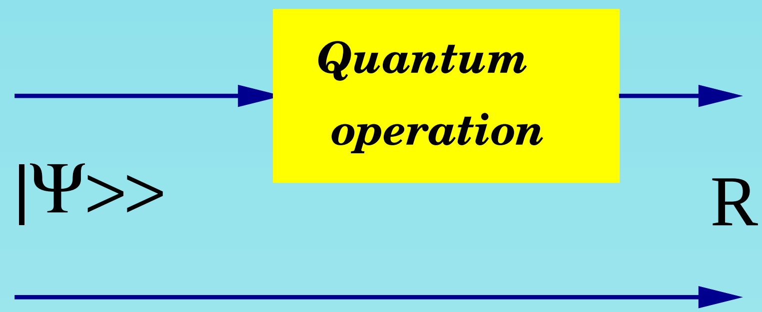

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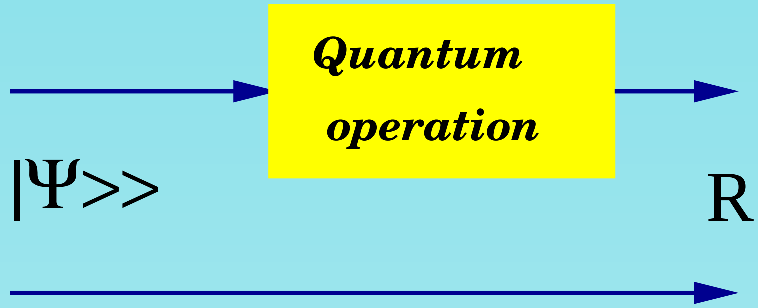

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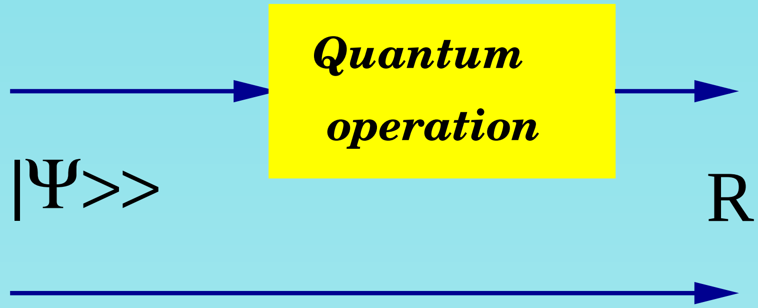

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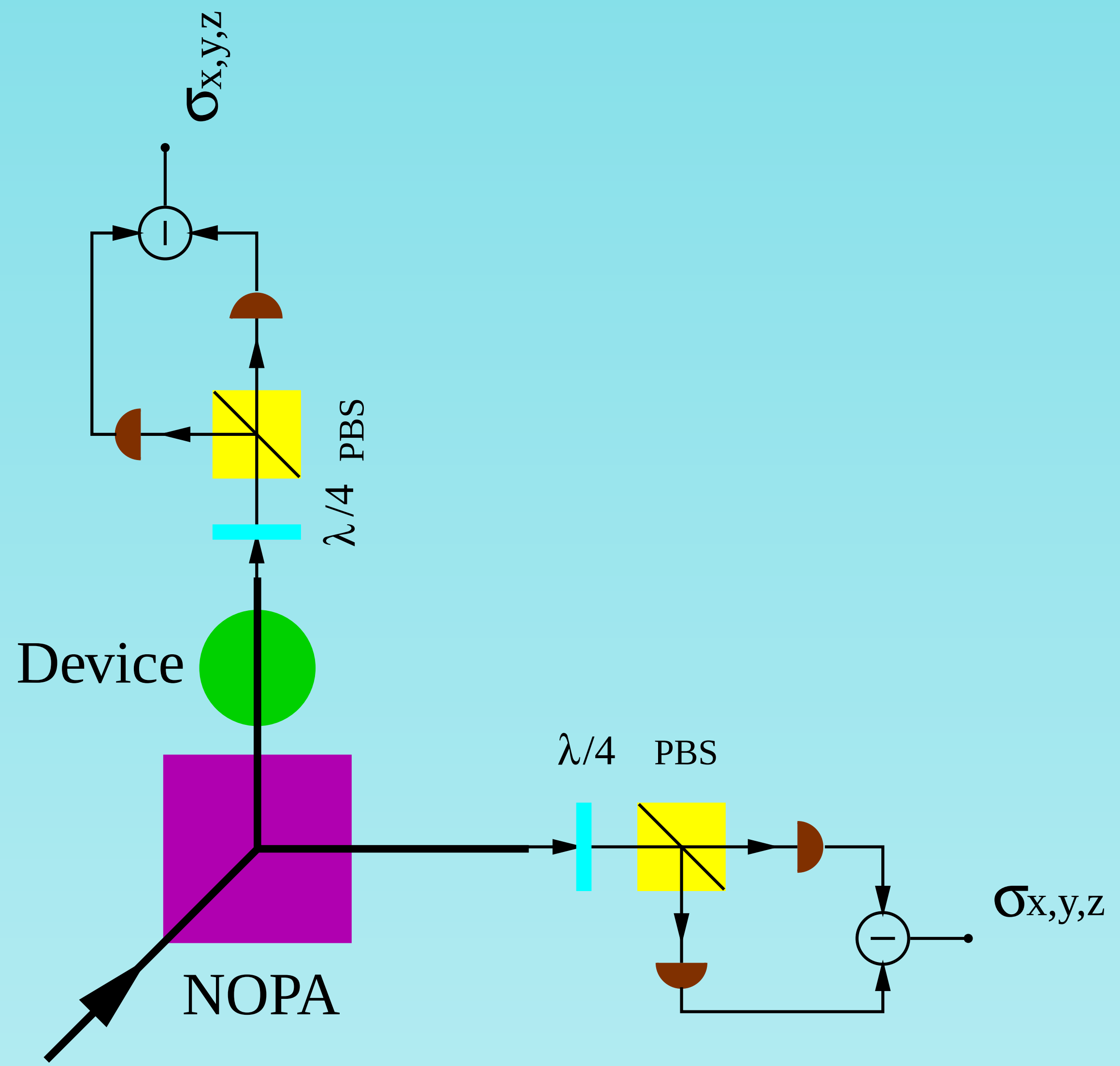
$\{|I\rangle\rangle, |\Psi\rangle\rangle\} \simeq$ choice of the representation for $\mathscr{E}$.

Tomography of a single qubit quantum device



[goto Pauli tomography]

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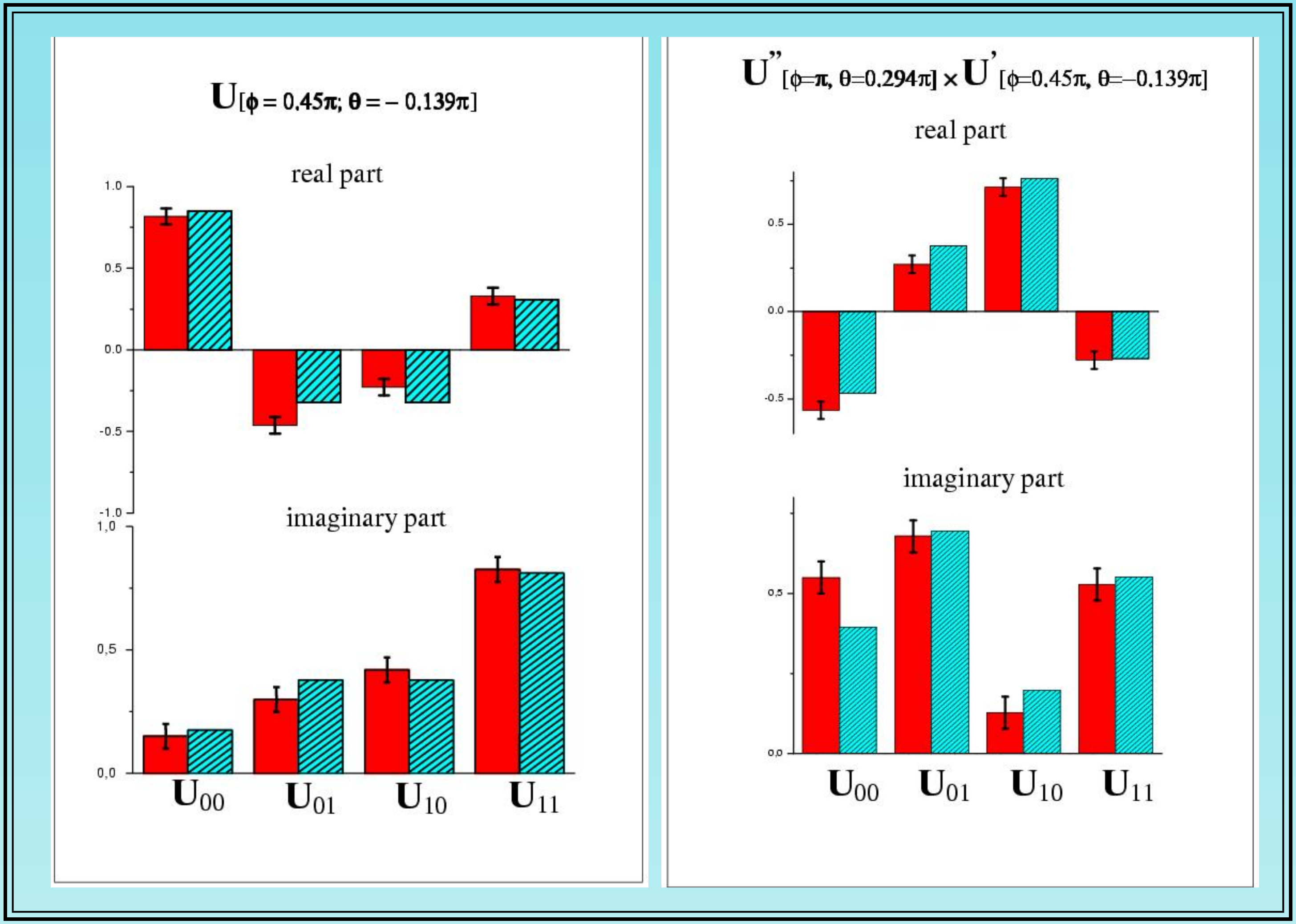
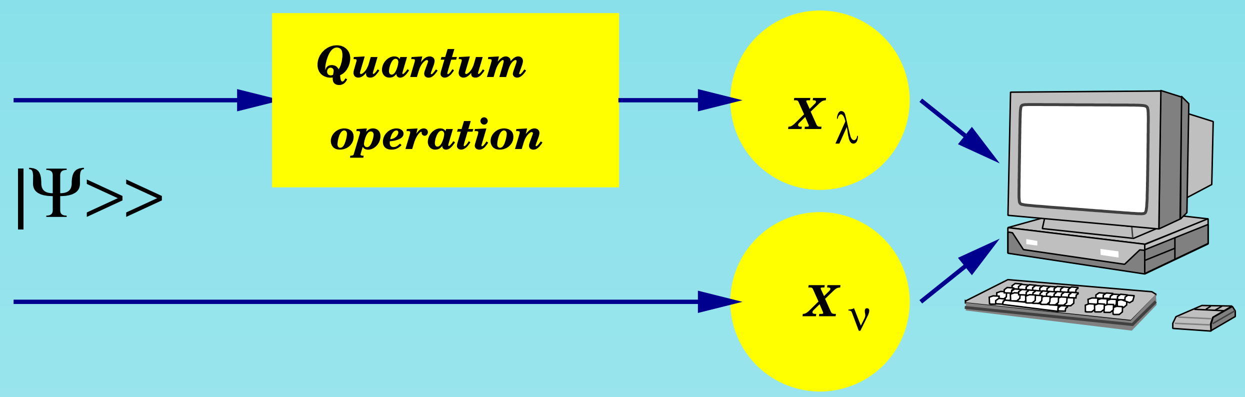
Experiment performed in Roma La Sapienza



Tomography of quantum operations



The QO (four-index matrix) $R_{\mathscr{E}}$ of the device is obtained by estimating via quantum tomography the following output ensemble averages

$$\langle\langle i, j | R_{\mathscr{E}} | l, k \rangle\rangle = \left\langle \; |l\rangle\langle i| \otimes \Psi^{-1*} |k\rangle\langle j| \Psi^{-1^{\tau}} \; \right\rangle . \tag{15}$$

[D'Ariano and Lo Presti, Phys. Rev. Lett. **86** *4195 (2001)]*

Faithful states

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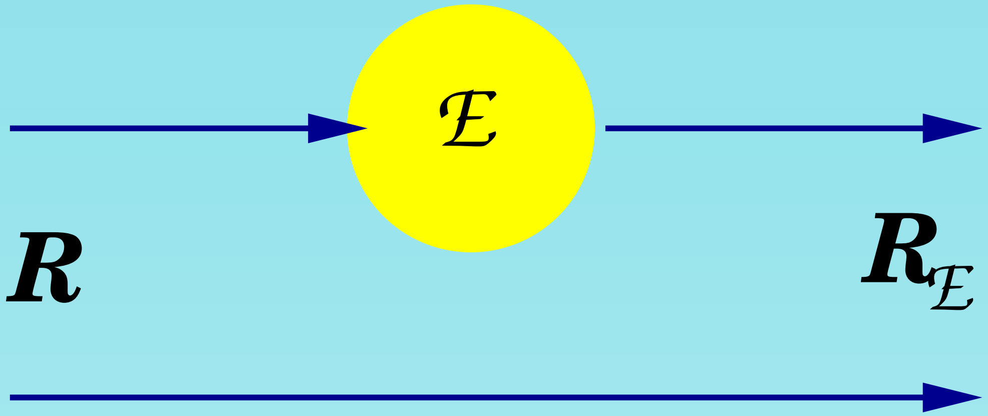
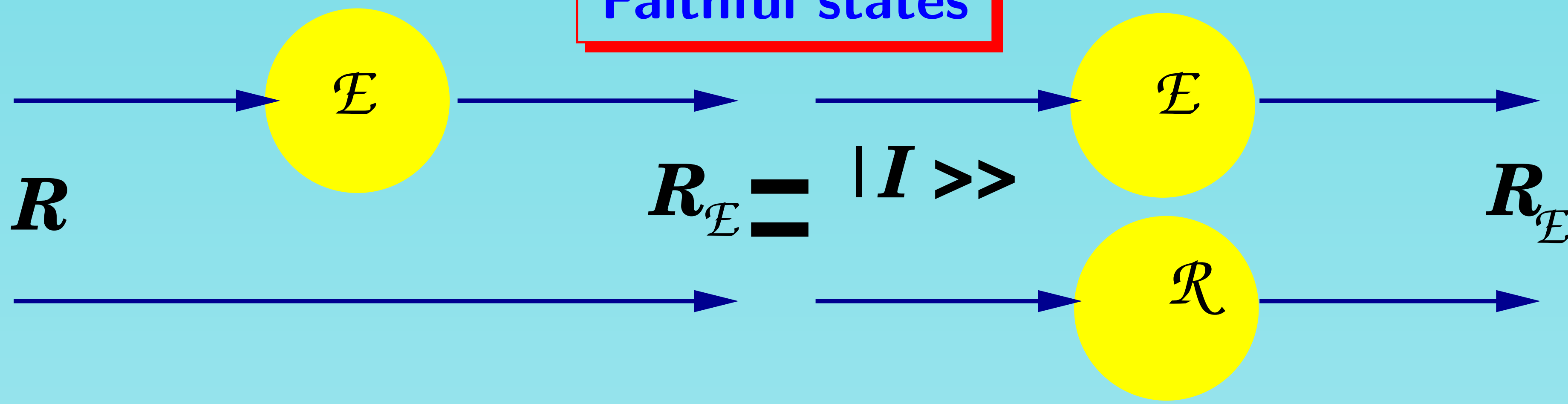
- We call a bipartite state *faithful* when acting with a channel on one of the two quantum systems, the output state carries a complete information about the channel.



$$R_{\mathscr{E}} \doteq \mathscr{E} \otimes \mathscr{I}(R). \tag{16}$$

Namely: the input state R is called *faithful* when the correspondence between the output state $R_{\mathscr{E}} \doteq \mathscr{E} \otimes \mathscr{I}(R)$ and the quantum channel $\mathscr{E}$ is one-to-one.

Faithful states



$$R = \sum_l |A_l\rangle\rangle\langle\langle A_l| = \mathscr{I} \otimes \mathscr{R}(|I\rangle\rangle\langle\langle I|), \; \mathscr{R}(\rho) = \sum_l A_l^\tau \rho A_l^*. \tag{17}$$

- A state R is faithful when it can be obtained from the maximally entangled vector with a map $\mathscr{R}$ that is invertible, in order to guarantee the one-to-one correspondence between $R_{\mathscr{E}}$ and $\mathscr{E}$.

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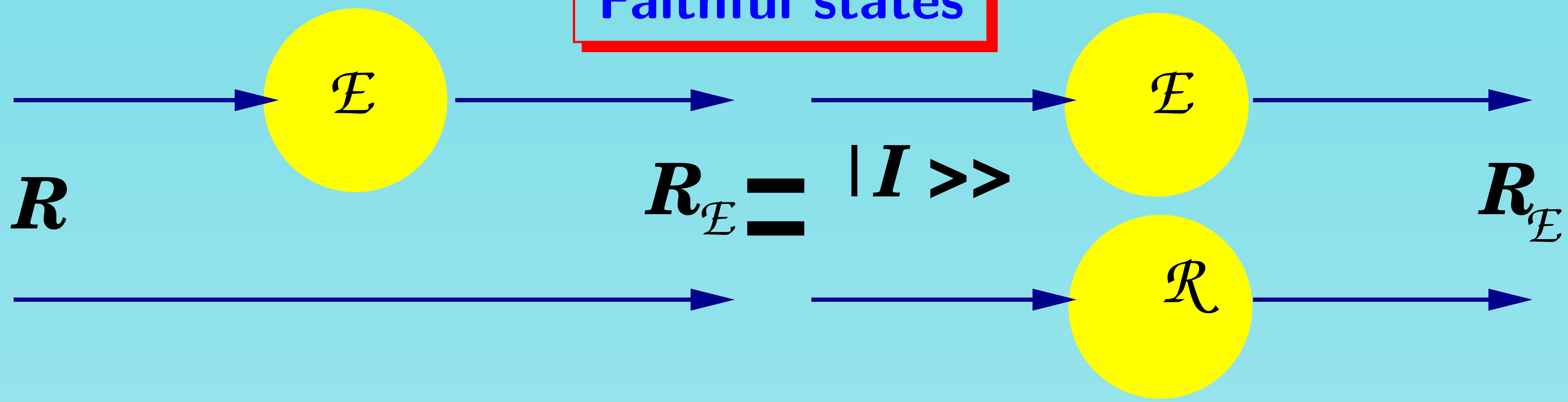

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[G. M. D'Ariano and P. Lo Presti, quant-ph/0211133]

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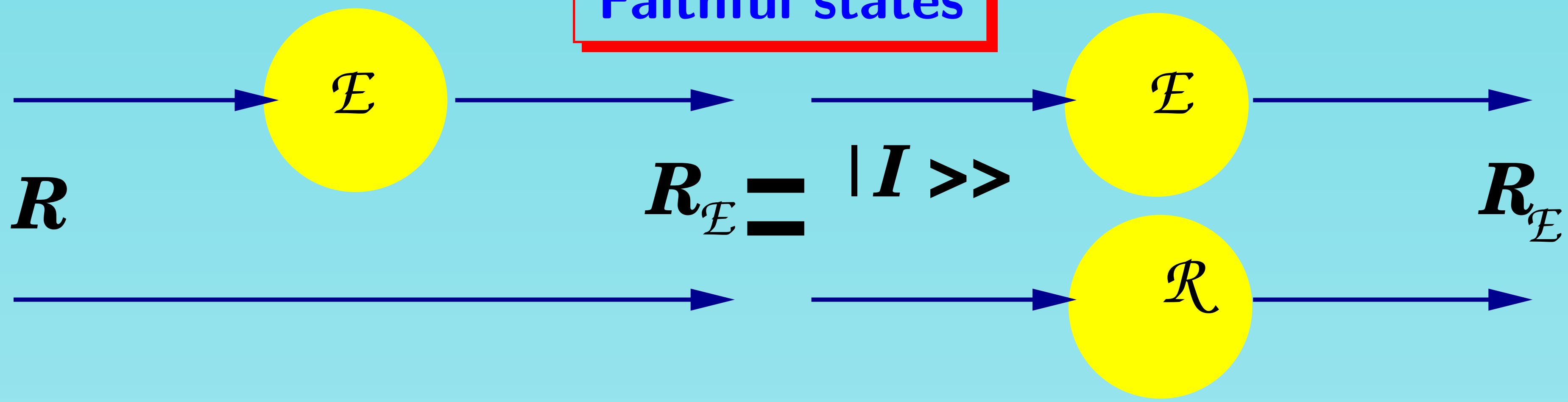

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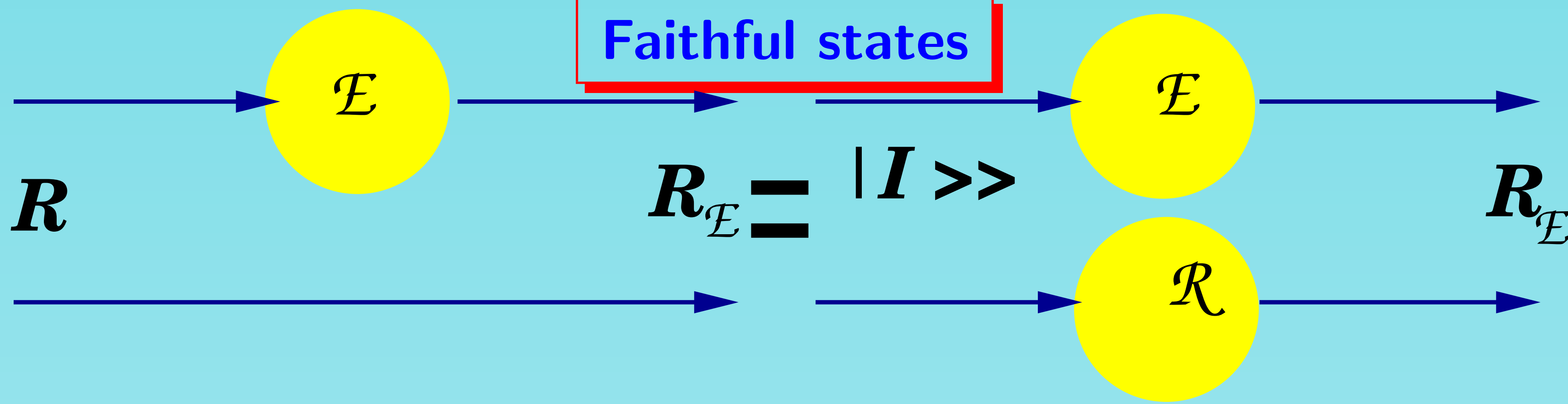
- A pure state $R \equiv |A\rangle\!\rangle\langle\!\langle A|$ is faithful iff it has maximal Schmidt's number.

Faithful states



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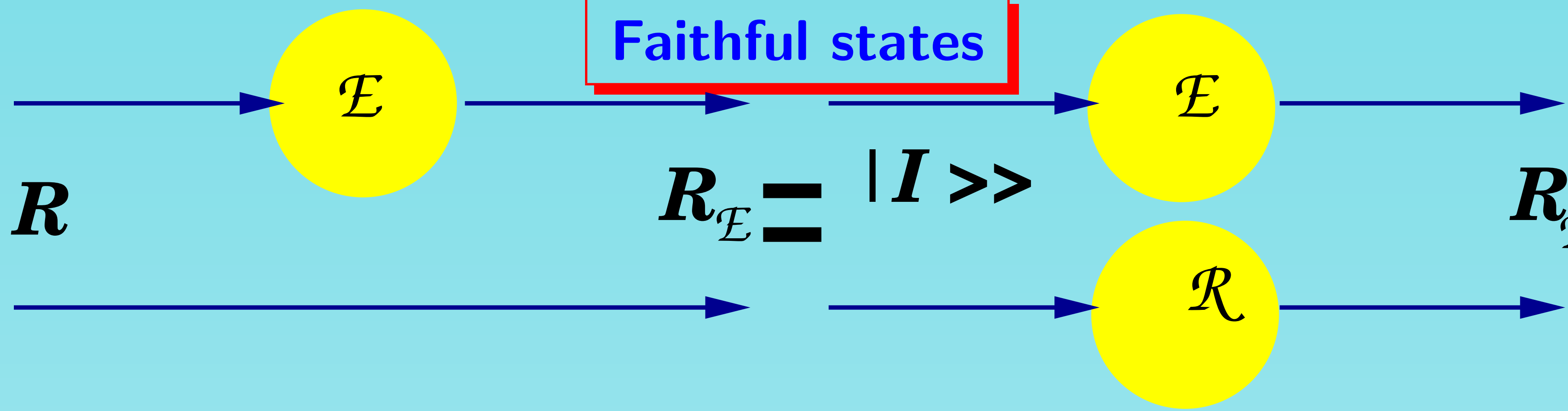

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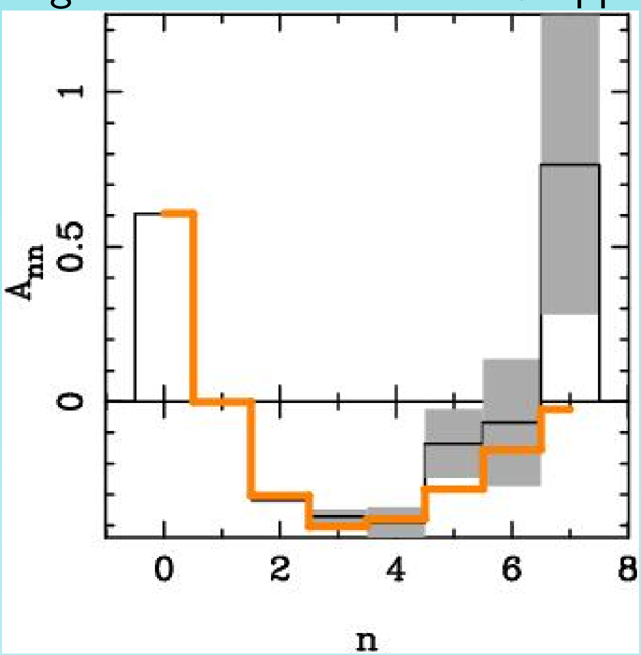

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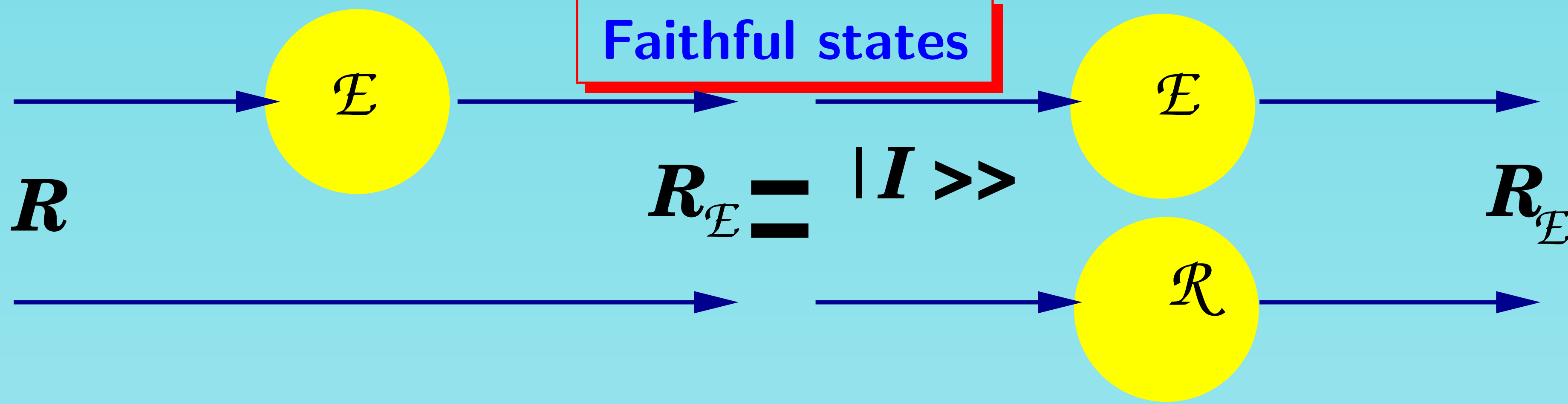

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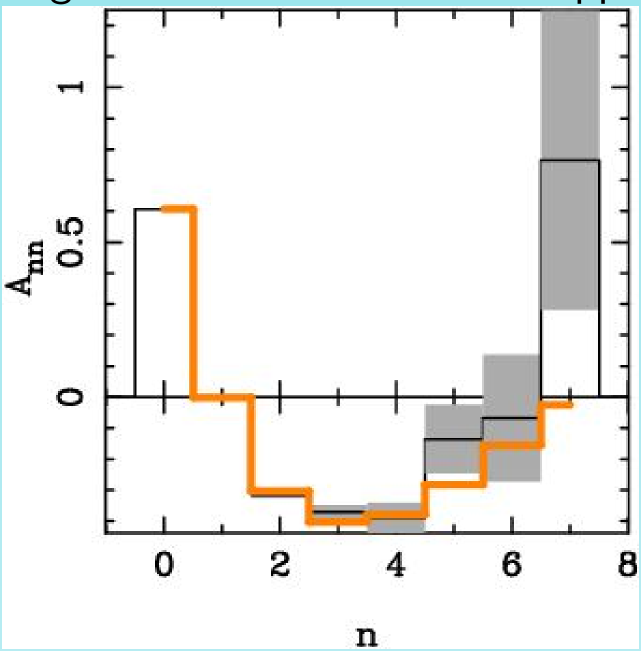
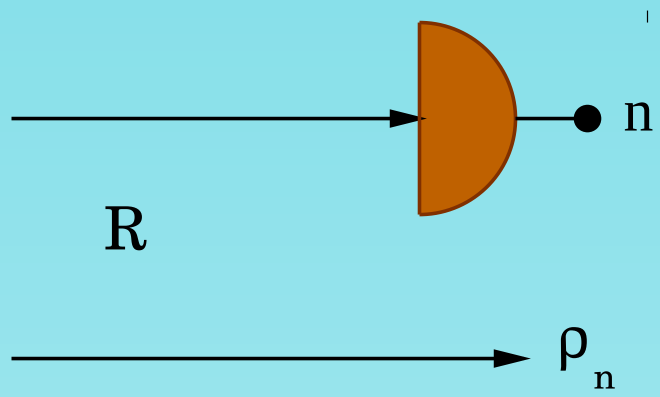


- Therefore, most mixed separable states are faithful! [e. g. Werner states are a. a. faithful].
- For c. v. faithfulness depends also on the matrix representation [e. g. Gaussian noise with $\overline{n} > \frac{1}{2}$].

Absolute Quantum Calibration: Tomography of POVM's



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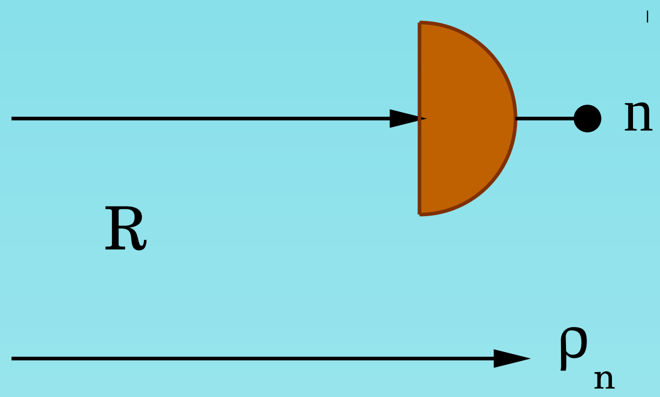

In terms of the POVM $\mathbf{P} \doteq (P_n)$ of the detector, the outcome n will occur with probability $p(n)$ corresponding to the conditioned state ρ_n given by

$$p(n) = \mathrm{Tr}[(P_n \otimes I)R], \qquad \rho_n = \frac{\mathrm{Tr}_1[(P_n \otimes I)R]}{\mathrm{Tr}[(P_n \otimes I)R]}, \tag{19}$$

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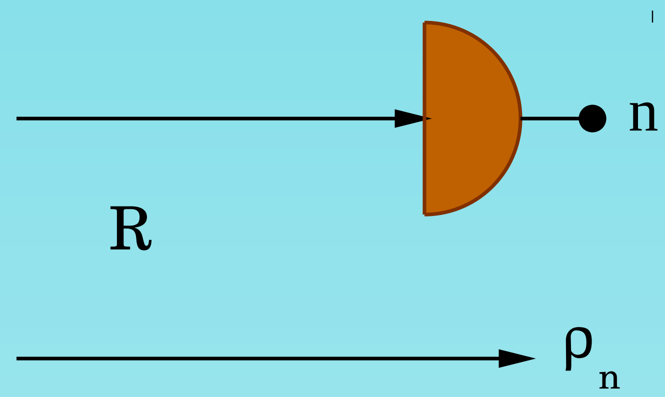

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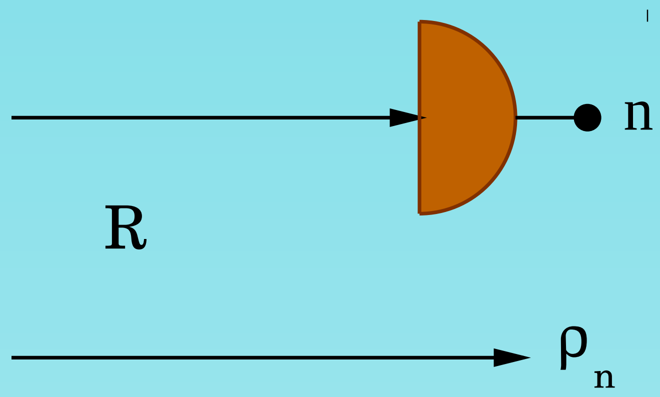
from which we can obtain the POVM as follows

$$P_n = p(n)[\mathscr{R}^{-1}(\rho_n)]^\tau, \quad \mathscr{R}(\rho) = \mathrm{Tr}_1[(\rho^\tau \otimes I)R]. \tag{20}$$

Absolute Quantum Calibration of Observable



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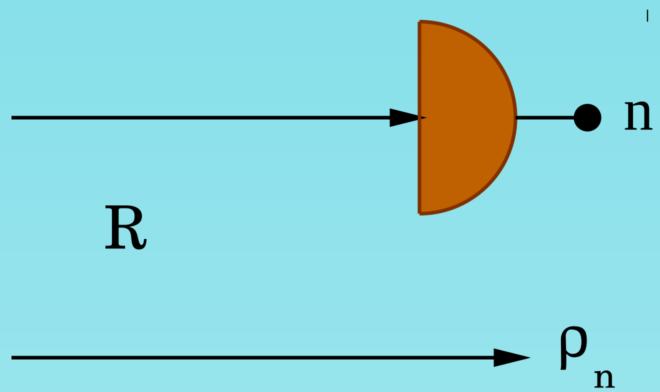

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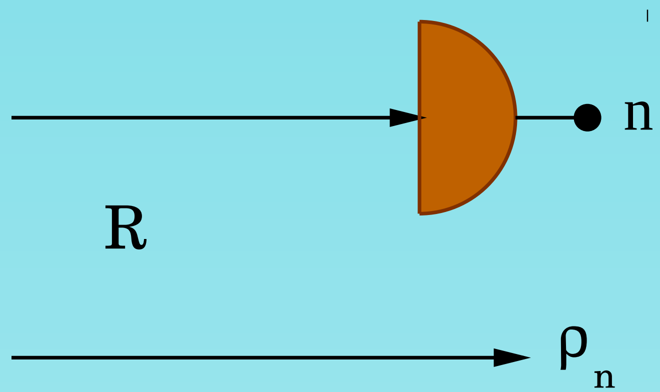

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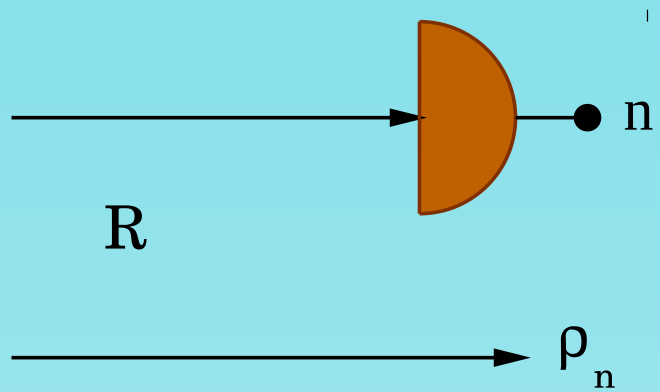

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  From tomographic data one reconstruct the matrix elements $\langle k|P_n|k\rangle = p(n|k)$ corresponding to the conditioned probability distribution $p(n|k)$.

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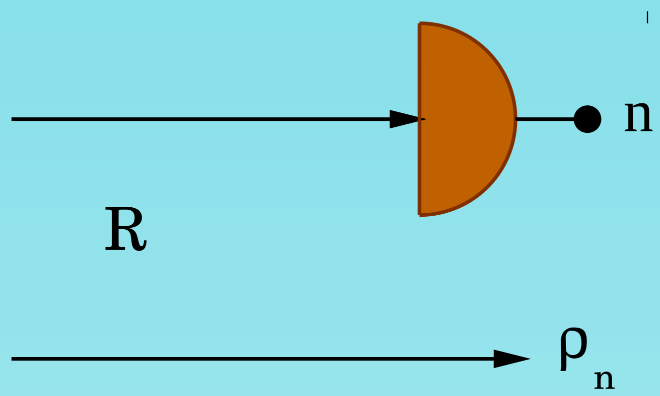
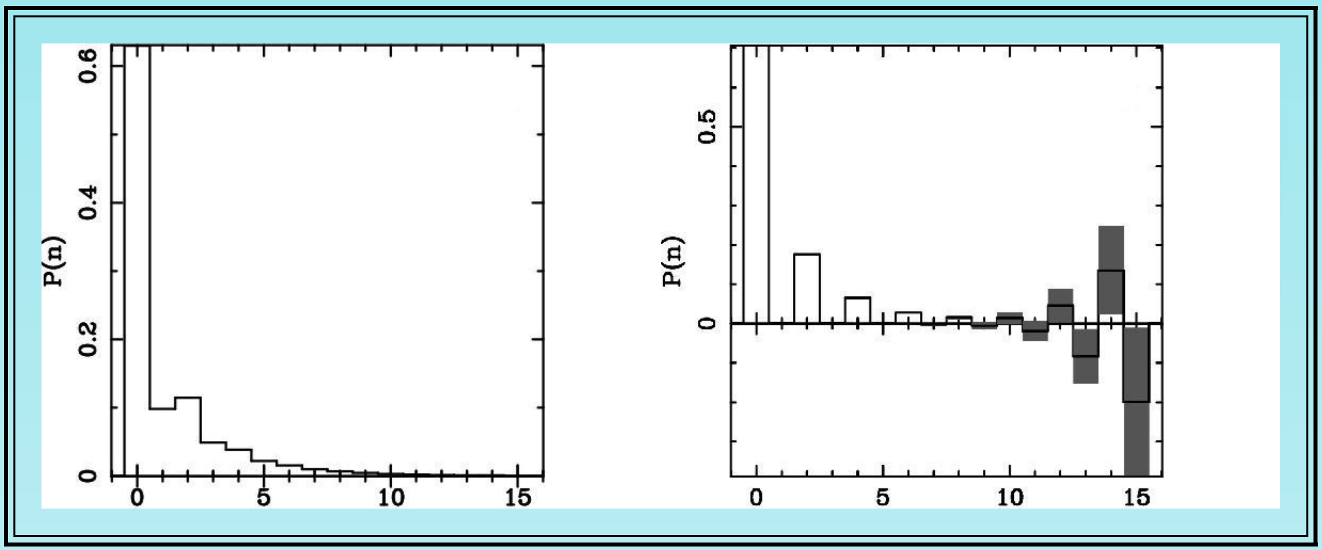


- The conditioned probability $p(n|k)$ from the tomographic calibration will allow "unbiasing" the detector measurements.



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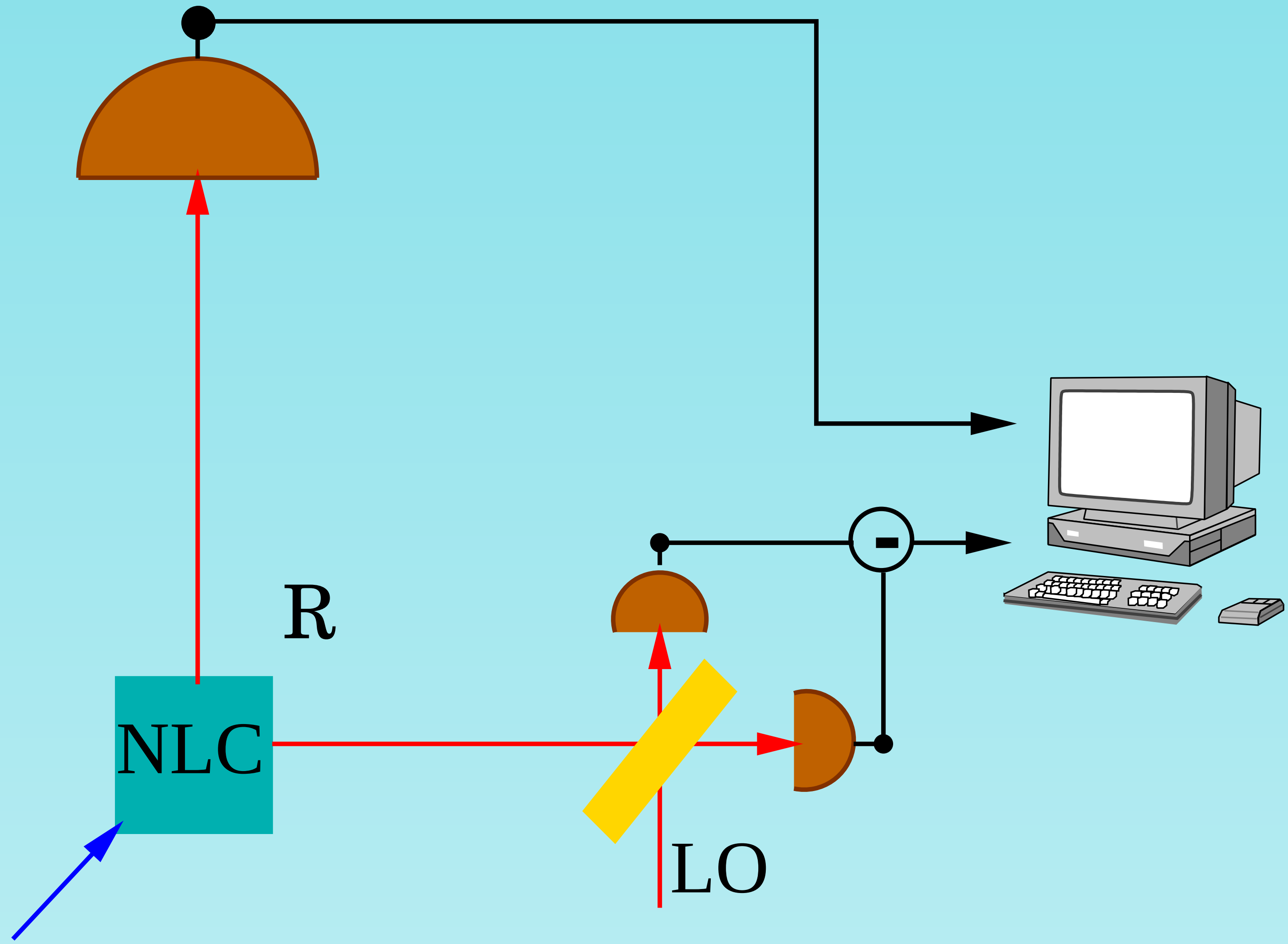

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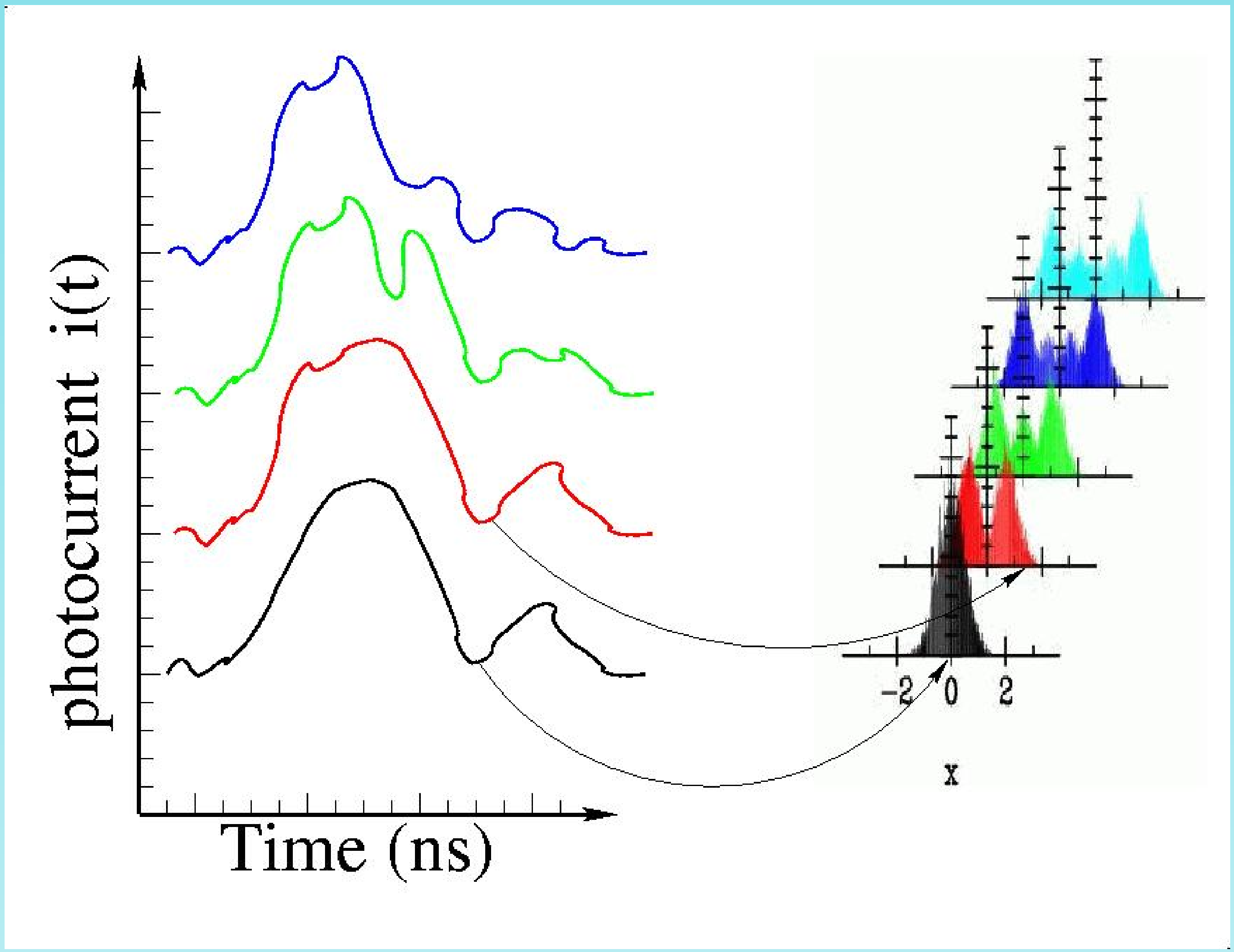

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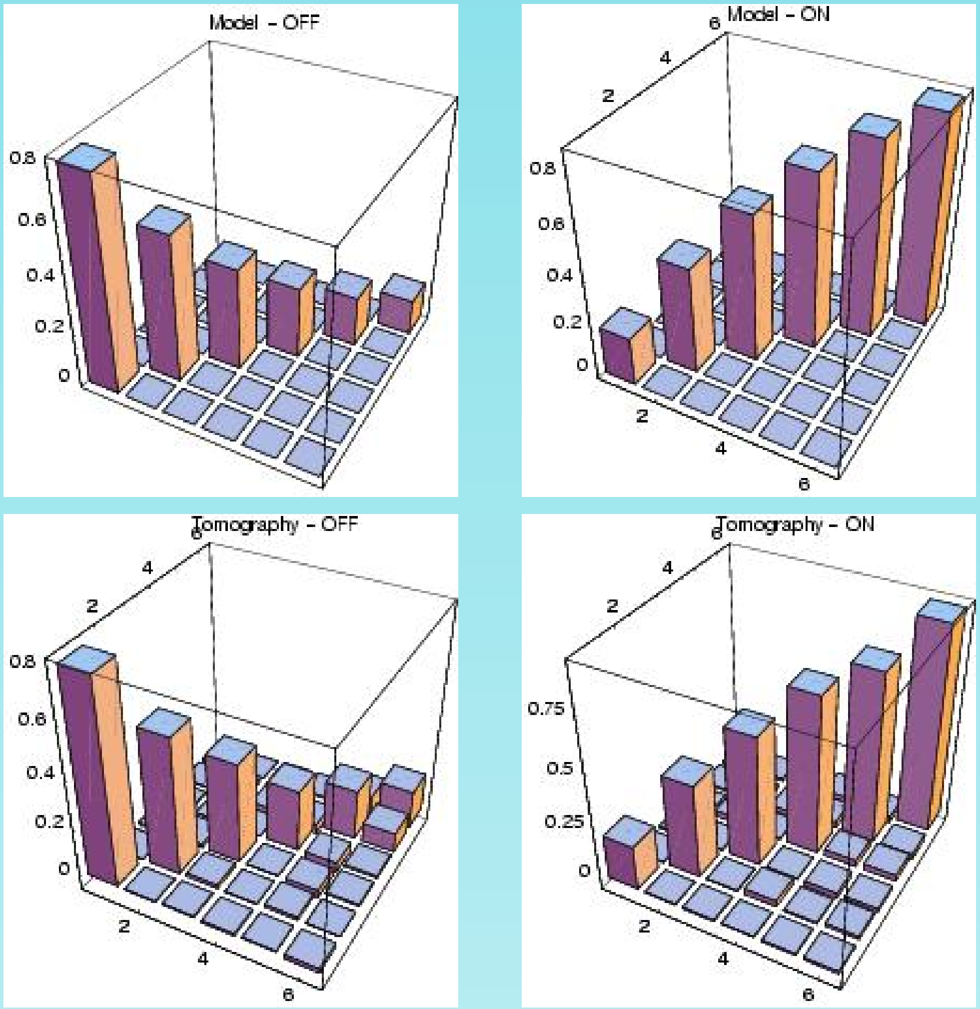

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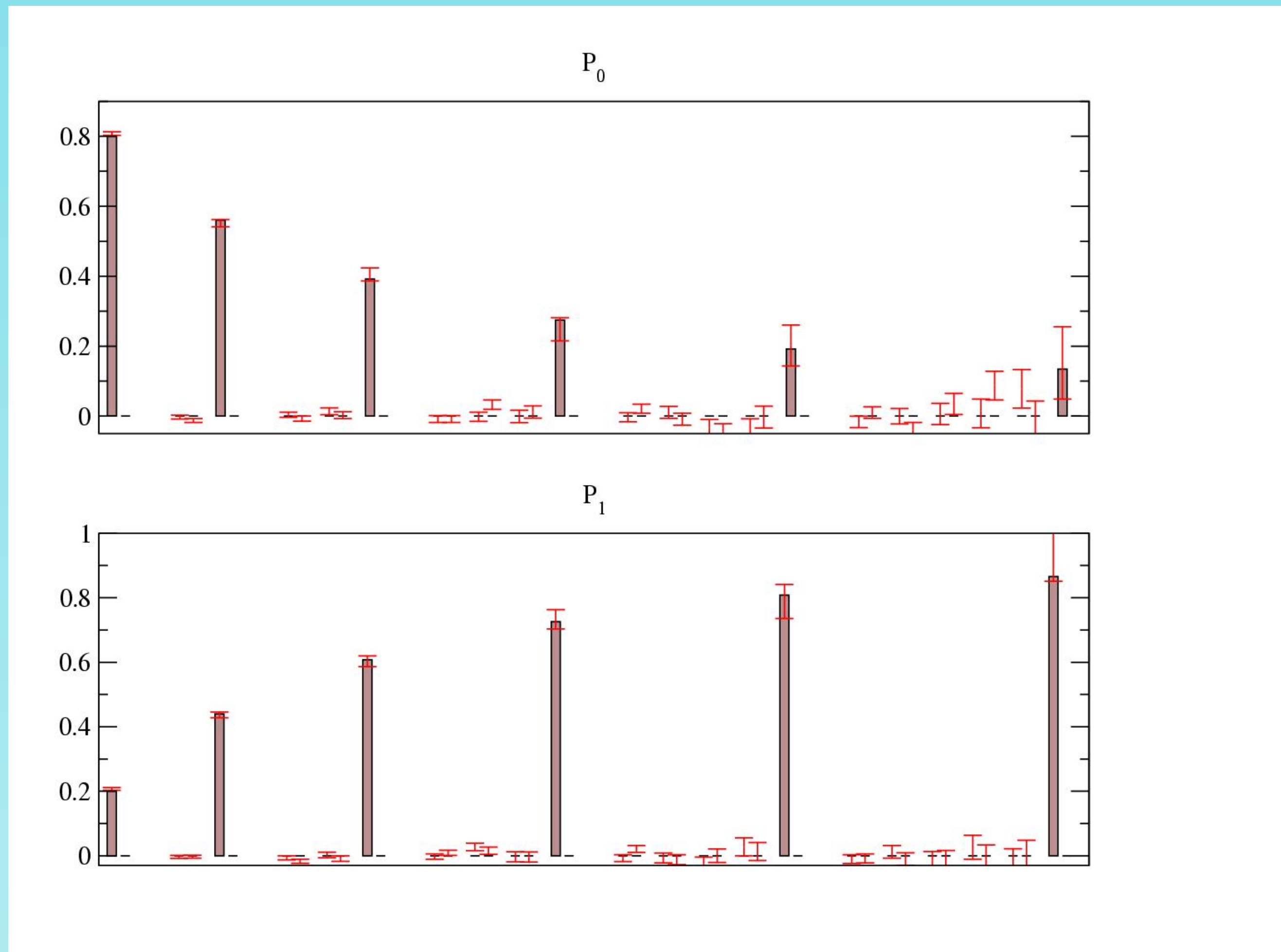

Computer simulation for 400.000 homodyne data, homodyne quantum efficiency $\eta = .8$ and $\overline{n} \simeq 4$ in the twin beam. [See NWU experiment]

[G. M. D'Ariano and P. Lo Presti, unpublished]

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Subject Index

What is a POVM?

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and the Born rule is given by

$$p(B) = \mathrm{Tr}[P_B \rho]. \tag{25}$$

[back to Tomography of POVM's]

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- The normalization $\mathrm{Tr}[\mathscr{E}(\rho)] \leq 1$ is the probability that the transformation occurs.

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4. Deterministic transformations (channels):
$$\mathrm{Tr}[\mathscr{E}(\rho)] = \mathrm{Tr}[\rho] \Rightarrow \sum_n K_n^\dagger K_n = I. \tag{30}$$

[back to tomography of quantum operation]

Complete positivity: relevant theorems

One-to-one correspondence $\mathscr{E} \leftrightarrow R_{\mathscr{E}}$ between quantum operations on $\mathsf{T}(\mathsf{H})$ and positive operators $R_{\mathscr{E}}$ on $\mathsf{H} \otimes \mathsf{H}$:

$$\begin{aligned} R_{\mathscr{E}} &= \mathscr{E} \otimes \mathscr{I}_{\mathsf{H}}(|I\rangle\rangle\langle\langle I|), \\ \mathscr{E}(\rho) &= \mathrm{Tr}_2[(I \otimes \rho^{\tau})R_{\mathscr{E}}], \end{aligned} \tag{31}$$

where

$$|I\rangle\rangle = \sum_n |n\rangle \otimes |n\rangle, \;\; \{|n\rangle\} \text{ orthonormal basis} \tag{32}$$

The most general form for $\mathscr{E}$ is (Kraus)

$$\mathscr{E}(\rho) = \sum_n K_n \rho K_n^{\dagger}, \tag{33}$$

where the operators K_n satisfy the bound

$$\sum_n K_n^{\dagger} K_n \le I. \tag{34}$$

[back to QO's]

Quantum tomography

- Quantum tomography is a method to estimate the ensemble average $\langle O \rangle$ of any arbitrary operator O on H by using only measurement outcomes of a *quorum* of observables $\{F_l\}$.

- The density matrix element ρ_{ij} corresponds to estimating the ensemble averages of $O = |j\rangle\langle i|$.

[G. M. D'Ariano, Scuola "E. Fermi", (IOS Press, Amsterdam 2002) pag. 385]

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- The tomographic estimation of the ensemble average $\langle O \rangle$ is obtained as double averaging over both the ensemble and the quorum.

- The method is very robust to all kinds of instrumental noises (general approach for unbiasing noise).

- It can be improved via "adaptive" techniques, maximum-likelihood strategies, etc.

- For multipartite quantum systems, simply a quorum is the tensor product of single-system quorums, namely one just needs to make local quorum measurements jointly on the subsystems.[back to tomography of QO's]

Pauli Tomography

Pauli matrices with identity I , σ_x , σ_y , σ_z: orthonormal basis for the qubit operator space:

$$H = \frac{1}{2}\{\boldsymbol{\sigma} \cdot \mathrm{Tr}[\boldsymbol{\sigma} H] + I\, \mathrm{Tr}[H]\}. \qquad (36)$$

Tomographic estimation:

$$\langle H \rangle = \frac{1}{3} \sum_{\alpha=x,y,z} \langle E_H(\sigma_\alpha;\alpha)\rangle, \qquad E_H(\sigma_\alpha;\alpha) = \frac{3}{2}\mathrm{Tr}[H\sigma_\alpha]\sigma_\alpha + \frac{1}{2}\mathrm{Tr}[H] \qquad (37)$$

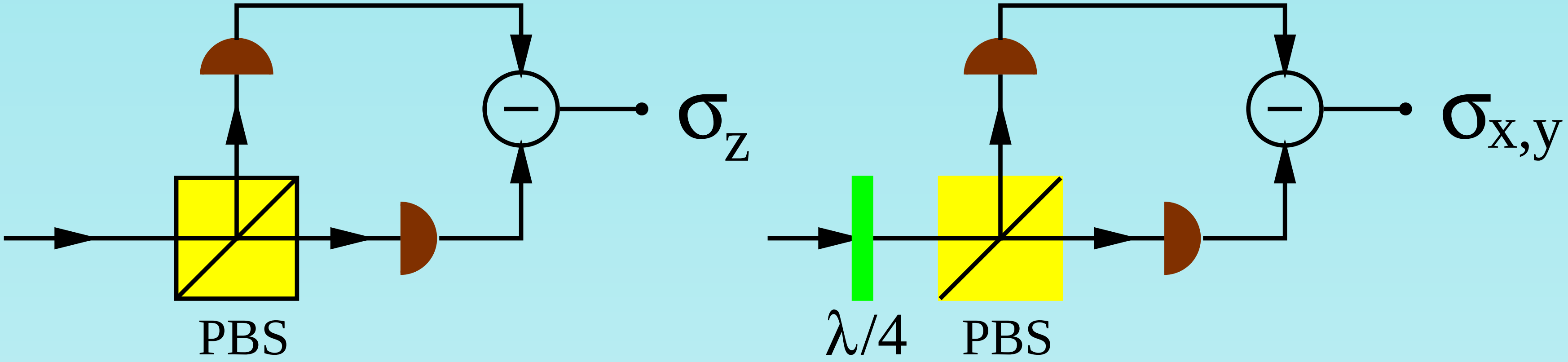
Pauli Tomography

Qubit realized by polarization of single photon states. [back to experiment]

$$\sigma_z = h^\dagger h - v^\dagger v, \qquad |\uparrow\rangle, \qquad |1\rangle_h \equiv |0\rangle_v, \quad |\downarrow\rangle \equiv |0\rangle_h |1\rangle_v, \tag{38}$$

$$\sigma_y = e^{i\frac{\pi}{4}\sigma_x} \sigma_z e^{-i\frac{\pi}{4}\sigma_x}, \qquad \sigma_x = e^{-i\frac{\pi}{4}\sigma_y} \sigma_z e^{i\frac{\pi}{4}\sigma_y}, \tag{39}$$

$$\begin{aligned} e^{-i\frac{\pi}{4}\sigma_x} |1\rangle_h |0\rangle_v &= \frac{1}{\sqrt{2}} [|1\rangle_h |0\rangle_v - i|0\rangle_h |1\rangle_v] \equiv |1\rangle_l |0\rangle_r, \\ e^{i\frac{\pi}{4}\sigma_y} |1\rangle_h |0\rangle_v &= \frac{1}{\sqrt{2}} [|1\rangle_h |0\rangle_v - |0\rangle_h |1\rangle_v] \equiv |1\rangle_{\nearrow} |0\rangle_{\nwarrow}. \end{aligned} \tag{40}$$



Homodyne tomography

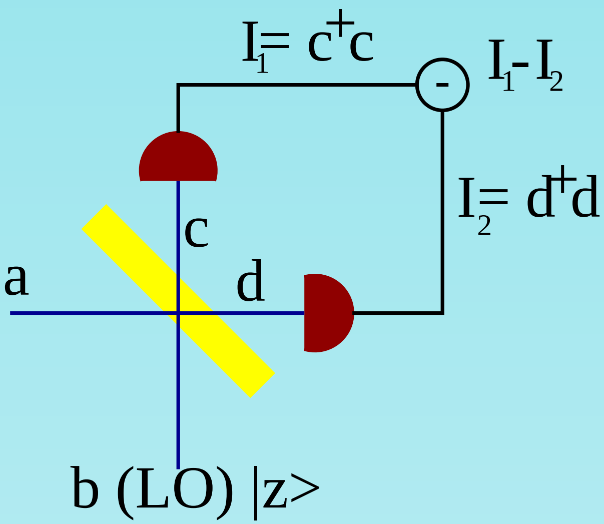
quadratures

In quantum optics a *quorum* for each mode of the field is given by the set of **quadratures**

$$X_\phi = \frac{1}{2}\left(a^\dagger e^{i\phi} + a e^{-i\phi}\right) . \quad (41)$$

One has

$$\langle H\rangle = \int_0^\pi \frac{\mathrm{d}\,\phi}{\pi}\langle E_H(X_\phi;\phi)\rangle, \quad E_H(x;\phi) = \frac{1}{4}\int_{-\infty}^{+\infty}\mathrm{d}\,k\,|k|\mathrm{Tr}[H e^{ikX_\phi}]e^{-ikx}, \quad (42)$$



$$c = \frac{1}{\sqrt{2}}(a+b)\,, \quad d = \frac{1}{\sqrt{2}}(a-b)\,, \quad (43)$$

$$I_D = I_1 - I_2 = a^\dagger b + b^\dagger a \simeq 2|z|X_\phi . \quad (44)$$

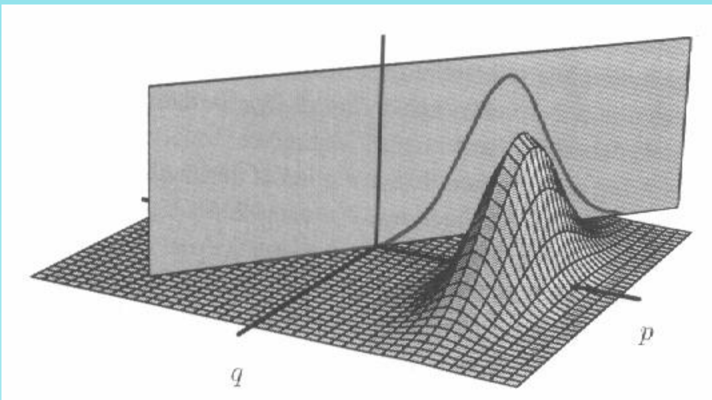
In the *strong LO limit* ($z \to \infty$) a balanced homodyne detector measures the quadrature X_ϕ of the field at any desired phase ϕ with respect to the local oscillator (LO).

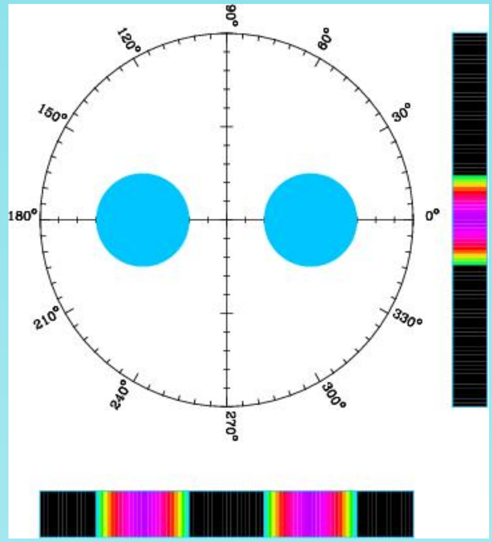
[G. M. D'Ariano, Scuola "E. Fermi", (IOS Press, Amsterdam 2002) pag. 385]

Homodyne tomography

$$\langle H \rangle = \int_0^\pi \frac{\mathrm{d}\,\phi}{\pi} \langle E_H(X_\phi;\phi)\rangle, \quad E_H(x;\phi) = \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}\,k\,|k|\mathrm{Tr}[H e^{ikX_\phi}] e^{-ikx}, \tag{45}$$

- Analogy with the **Radon transform** for *imaging*





- A **tomography** of a two dimensional image $W(\alpha,\bar{\alpha})$ is a collection of one dimensional projections $p(x;\phi)$ at different values of the observation angle ϕ.

$$W(\alpha,\overline{\alpha}) = \int_{-\infty}^{+\infty} \frac{dr|r|}{4} \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx\, p(x;\phi) \exp\left[ir(x-\alpha_\phi)\right], \tag{46}$$

[back to tomography of quantum operation]

Entangled states

Entangled states $|\Psi\rangle\rangle \in \mathsf{H} \otimes \mathsf{H}$

$$|\Psi\rangle\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle. \tag{47}$$

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Matrix notation (for fixed reference basis in the two Hilbert spaces):

$$A \otimes B|C\rangle\rangle = |AC\,B^\tau\rangle\rangle, \qquad (48)$$

$$|A\rangle\rangle \doteq \sum_{nm} A_{nm}|n\rangle \otimes |m\rangle \equiv A \otimes I|I\rangle\rangle \equiv I \otimes A^\tau|I\rangle\rangle, \qquad (49)$$

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Isomorphism $\mathsf{HS}(\mathsf{H}) \simeq \mathsf{H} \otimes \mathsf{H}$ between the Hilbert space $\mathsf{HS}(\mathsf{H})$ of Hilbert-Schmidt operators on H and $\mathsf{H} \otimes \mathsf{H}$

$$\langle\langle A|B\rangle\rangle \equiv \mathrm{Tr}[A^\dagger B]. \qquad (51)$$

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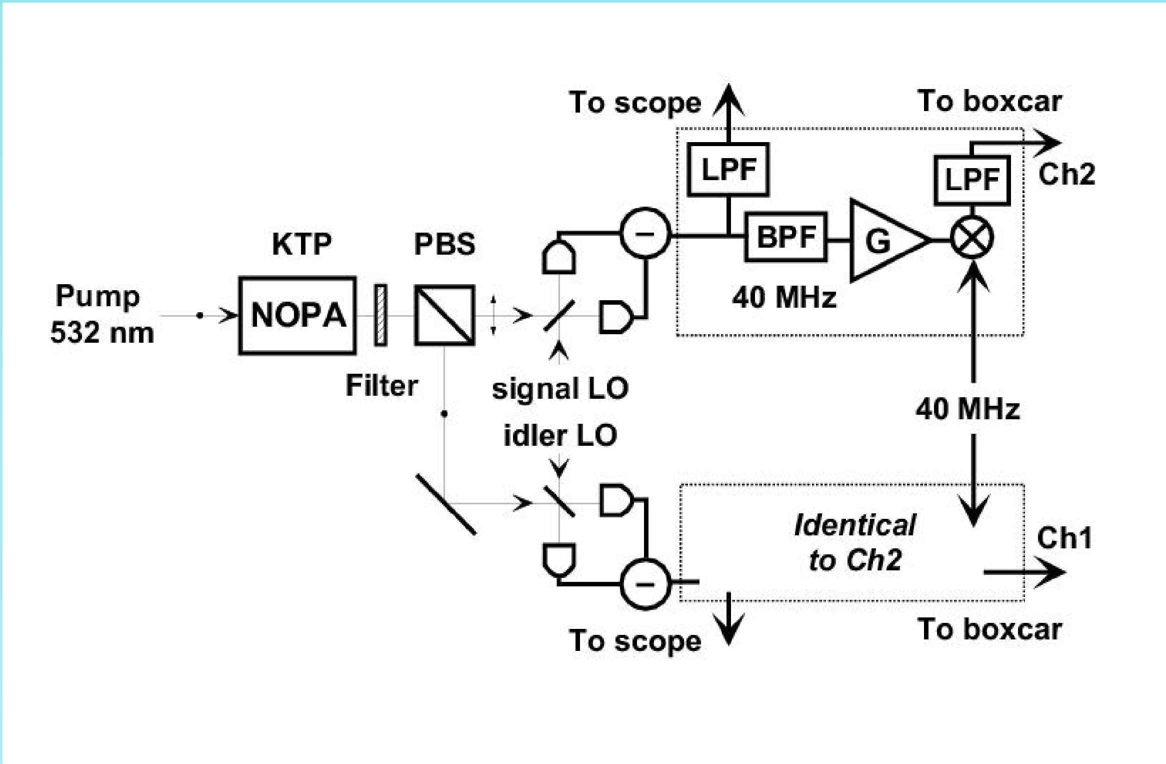
Measure of the entanglement for pure states: von Neumann entropy $S(\rho) = -\,\mathrm{Tr}[\rho \ln \rho]$ of the local state

$$\rho = \mathrm{Tr}_2[|\Psi\rangle\rangle\langle\langle\Psi|] \equiv \Psi\Psi^\dagger. \tag{52}$$

[back to tomography of QO's]

Some experimental results

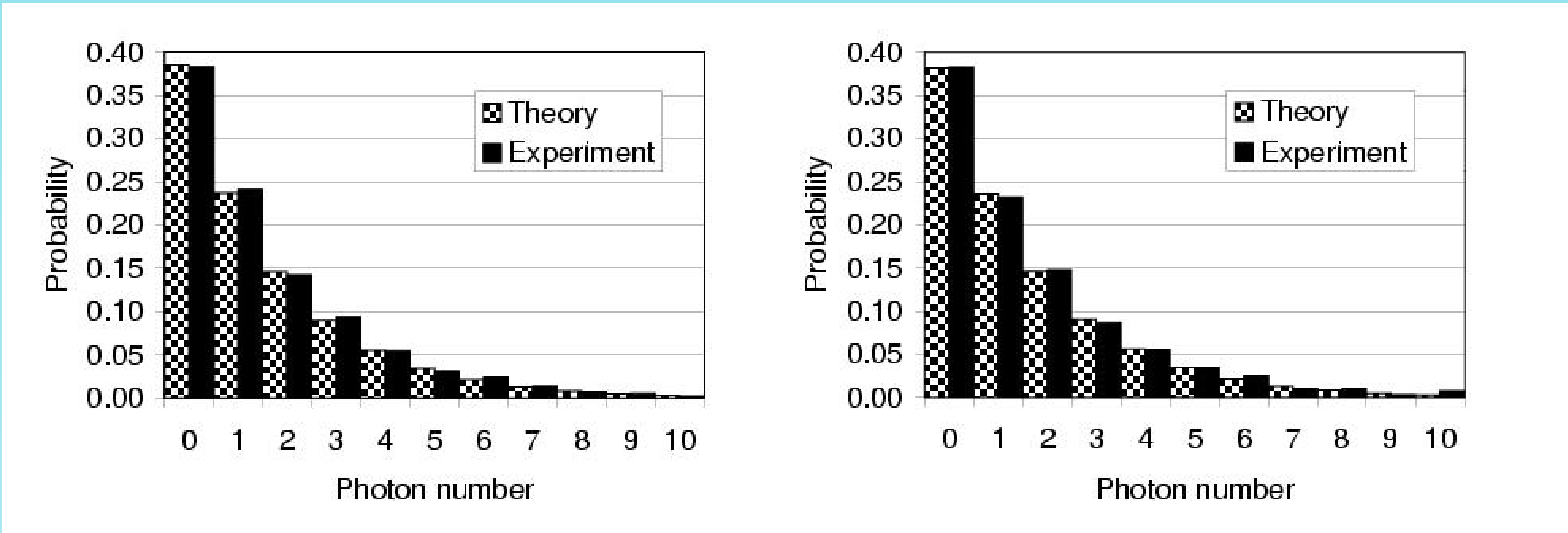
A schematic of the experimental setup. NOPA, non-degenerate optical parametric amplifier; LOs, local oscillators; PBS, polarizing beam splitter; LPFs, low-pass filters; BPF, band-pass filter; G, electronic amplifier. Electronics in the two channels are identical. The measured distributions exhibit up to 1.9 dB of quantum correlation between the signal and idler photon numbers, whereas the marginal distributions are thermal as expected for parametric fluorescence.



Measurement of the joint photon-number probability distribution for a twin-beam from nondegenerate downconversion

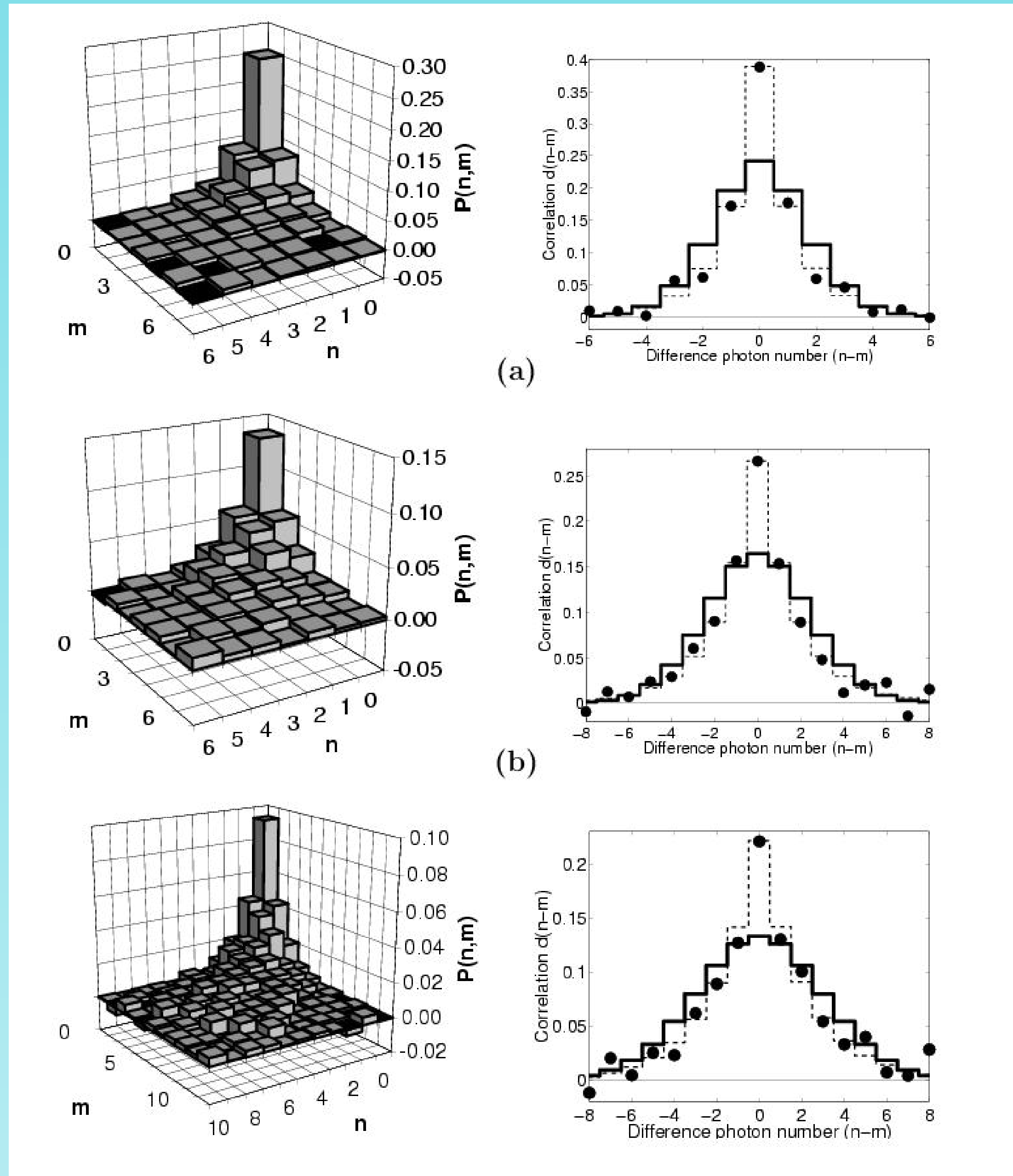
Some experimental results

Marginal distributions for the signal and idler beams. Theoretical distributions for the same mean photon numbers are also shown [Phys. Rev. Lett. **84** 2354 (2000)].



Results

Left: Measured joint photon-number probability distributions for the twin-beam state. Right: Difference photon number distributions corresponding to the left graphs (filled circles, experimental data; solid lines, theoretical predictions; dashed lines, difference photon-number distributions for two independent coherent states with the same total mean number of photons and $\overline{n} = \overline{m}$.) (a) 400000 samples, $\overline{n} = \overline{m} = 1.5$, $N = 10$; (b) 240000 samples, $\overline{n} = 3.2$, $\overline{m} = 3.0$, $N = 18$; (c) 640000 samples, $\overline{n} = 4.7$, $\overline{m} = 4.6$, $N = 16$. [back to photodetector calibration]



Examples of faithful states

- Werner's states:

$$R_f = \frac{1}{d(d^2-1)}[(d-f) + (df-1)E], \quad (53)$$

  E swap operator, $d = \dim(\mathsf{H})$, $(-1 \le f \le 1)$

- faithful for all $f \neq \frac{1}{d}$, separable for $f \ge 0$.

- Isotropic states for dimension d

$$R_f = \tfrac{f}{d}|I\rangle\!\rangle\langle\!\langle I| + \tfrac{1-f}{d^2-1}(I - \tfrac{1}{d}|I\rangle\!\rangle\langle\!\langle I|), \quad (54)$$

- faithful for $f \neq \frac{1}{d^2}$, separable for $f \le \frac{1}{d}$. [back to faithful states]

Faithful states for "continuous variables"

- The inverse map $\mathscr{R}^{-1}$ is unbounded.

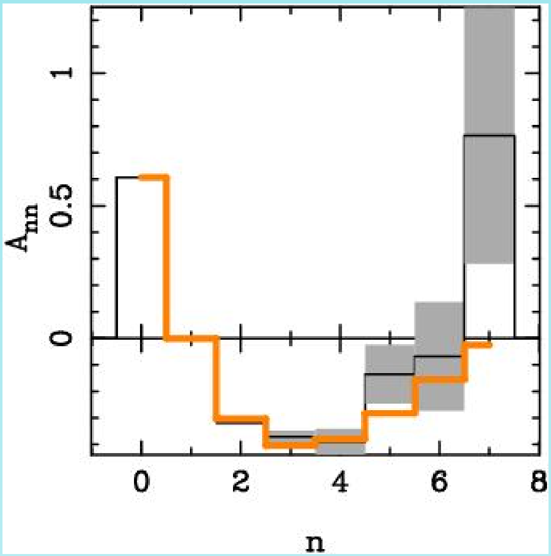
- As a result we will recover the channel $\mathscr{E}$ from the measured $R_{\mathscr{E}}$ with unbounded amplification of statistical errors, (depending on the chosen representation).

- Example: twin beam from parametric down-conversion of vacuum

$$|\Psi\rangle\rangle = \Psi \otimes I|I\rangle\rangle, \quad \Psi = (1-|\xi|^2)^{\frac{1}{2}}\xi^{a^\dagger a}, \qquad |\xi| < 1. \tag{55}$$

- The state is faithful, but the operator Ψ^{-1} is unbounded, whence the inverse map $\mathscr{R}^{-1}$ is also unbounded.

- For example, in a photon number representation $\mathsf{B} = \{|n\rangle\langle m|\}$, the effect will be an amplification of errors for increasing numbers n, m of photons.



Faithful states for "continuous variables"

- Consider now the quantum channel describing the *Gaussian displacement noise*

$$\mathscr{N}_\nu(\rho) = \int_{\mathbb{C}} \frac{\mathrm{d}\,\alpha}{\pi\nu} e^{-\frac{|\alpha|^2}{\nu}} D(\alpha)\rho D^\dagger(\alpha), \tag{56}$$

- analogous of the depolarizing channel for infinite dimension.

- From the multiplication rule $\mathscr{N}_\nu \mathscr{N}_\mu = \mathscr{N}_{\nu+\mu}$, it follows that the inverse map is formally given by

$$\mathscr{N}_\nu^{-1} \equiv \mathscr{N}_{-\nu}. \tag{57}$$

- As a faithful state consider now the mixed state given by the twin-beam, with one beam spoiled by the Gaussian noise, namely

$$R = \mathscr{I} \otimes \mathscr{N}_\nu(|\Psi\rangle\rangle\langle\langle\Psi|) = \frac{1}{\nu}(\Psi \otimes I)\exp\left[-\frac{(a-b^\dagger)(a^\dagger-b)}{\nu}\right](\Psi^\dagger \otimes I), \tag{58}$$

  The partial transposed is

$$R^{\tau_2} = (\nu+1)^{-1}(\Psi \otimes I)\left(\frac{\nu-1}{\nu+1}\right)^{\frac{1}{2}(a-b)^\dagger(a-b)}(\Psi^\dagger \otimes I), \tag{59}$$

- Since our state is Gaussian, the PPT criterion guarantees separability[R. Simon, Phys. Rev. Lett. **84**, 2726 (2000)] and for $\nu > 1$ our state is separable, still it is *formally* faithful, since the operator Ψ and the map N_ν are both invertible.

Faithful states for "continuous variables"

- Unboundedness of the inverse map can wash out completely the information on the channel in some particular chosen representation.
- Example: (overcomplete) representation $\mathsf{B} = \{|\alpha\rangle\langle\beta|\}$, with $|\alpha\rangle$ and $|\beta\rangle$ coherent states.

- From the identity

$$\mathcal{N}_\nu(|\alpha\rangle\langle\alpha|) = \frac{1}{\nu+1} D(\alpha) \left(\frac{\nu}{\nu+1}\right)^{a^\dagger a} D^\dagger(\alpha), \tag{60}$$

one obtains

$$\mathcal{N}_\nu^{-1}(|\alpha\rangle\langle\alpha|) = \frac{1}{1-\nu} D(\alpha) \left(1-\nu^{-1}\right)^{-a^\dagger a} D^\dagger(\alpha), \tag{61}$$

- which has convergence radius $\nu \leq \frac{1}{2}$, which is the well known bound for Gaussian noise for the quantum tomographic reconstruction for coherent-state and Fock representations.[1]
- Therefore, we say that the state is *formally* faithful, however, we are constrained to representations which are analytical for the inverse map $\mathscr{R}^{-1}$. [back to faithful states]

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[1] G. M. D'Ariano, and N. Sterpi, J. Mod. Optics **44** 2227 (1997)