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NON-ABELIAN QUANTUM WALKS AND RENORMALIZATION

Quantum Walk non abeliani e Rinormalizzazione

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ABSTRACT

Abstract

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INTRODUCTION

1 The world is everything that is the case.
1.1 The world is the totality of facts, not of things.

— LUDWIG WITTGENSTEIN [Tractatus Logico-Philosophicus]

Atomism is the principle of natural philosophy according to which everything is constituted by fundamental and indivisible bits of matter, called *atoms*. This idea was theorized by the ancient Greek philosopher Democritus and gave rise to a long-standing debate that went on for centuries.
The first attempt to formalise this principle in physics was made by Bernoulli in his treatise *Hydrodynamica*, which opened the way to the *kinetic theory of gases*. The subsequent development of *Statistical Mechanics* was a first, vivid example of how fundamental laws may lead to emergent behaviours.
In 1827, the botanist Robert Brown reported in [1] the empirical evidences of "the general existence of active molecules in organic and inorganic bodies" and contributed to open an academic discussion about the atomistic hypothesis. Einstein, in 1905 (the *Annus Mirabilis*), correctly explained the phenomenon in [2], furnishing a mathematical description of the microscopic mechanism underlying the motion of a particle suspended in a fluid. Such a model for the dynamical evolution of a system is precisely a *random walk* (RW).

RWs are ubiquitous in science and play an important role in many disciplines. One of their most interesting and relevant features is that RWs exhibit emergent complexity, despite the simplicity of the rules governing their evolution. This property is even more prominent when considering the *quantum* counterpart of

a classical random walk, namely the quantum walk (QW).
QWs have gained a well founded mathematical structure (see *e.g.* [3–8]) and aroused interest in *Computer Science* and in *Quantum Information*, since they proved to be an useful tool in designing efficient quantum algorithms [9–11].
QWs have been extensively taken into account in *lattice gauge theories* [12–16], furnishing an appropriate computational tool to overcome the severe mathematical problems that arose in *Quantum Field Theory* (QFT).
In his pioneering paper [17], Bialynicki-Birula realised the potential fundamentality held by QWs when applied to QFT: the author thus shed light on a possible intrinsic connection between the particular geometry of a lattice and free quantum fields' evolution.

Considering the description of free quantum fields provided by QFT as the emergent trait of a discrete fundamental mechanism is a very fascinating idea and entails serious implications, of both physical and philosophical nature. This is evocative of Wheeler's and Feynman's paradigm of *physics as information processing* [18].
The program of founding physical laws on informational principles has been recently considered by Giacomo Mauro D'Ariano *et al.* ; in [19] D'Ariano, Perinotti and Chiribella provide an axiomatization of finite-dimensional *Quantum Theory*, which is then *derived* from a set of informational axioms. In [20–27], the author proposed to take QWs as the fundamental physical law at Planck-scale, supporting the idea of a discrete spacetime (for both experimental and theoretical discussions about violations of Lorentz symmetry at Planck-scale, see *e.g.* [28–34]).
This framework is characterized by an *operational* perspective, as defined by Bridgman in [35]: "We mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations". Moreover, it leans on the *Deutsch-Church-Turing thesis* [36, 37], as rephrased in [22]: "Every finite experimental protocol is perfectly simulated by a finite quantum algorithm".
QFT is lacking of both an operational definition of the concept of *field* and of a rigorous *theory of measurement*[1]. This is a delicate matter; some have proposed to overcome these issues through the *informational paradigm*.
Appealing to first principles in physics is a suggestive idea, *a fortiori* in the perspective of reconsidering physics' basic tenets. A possible way to accomplish

1 In [38] Feynman argues: "There might be something wrong with the old concept of continuous functions. How could there possibly be an infinite amount of information in any finite volume?"

this program is questioning about the physical postulates which cannot be renounced.

The model presented, which is quantum *ab initio*, regards *quantum fields* as emergent from a network of countably many quantum system in mutual interaction, requiring a strict notion of *locality*. This fundamental emergence from interacting neighbouring sites has been taken into account as an attempt of going over the idea of *object* in physics, recovering a theory describing *events* (in the spirit of Wheeler's famous quote: "*It from bit*").
One of the consequences of considering a self-interacting quantum network is that the metric is derived from pure event-counting[2].

0.1 SCOPE AND CONTENTS

In this thesis work we will review the paper [40], in which QWS over Cayley graphs are extensively studied for abelian groups, giving a large-scale Dirac evolution from the informational and operational postulates aforementioned.
We will then propose a procedure that resorts to a group-theoretical treatment; it will be performed in the *position space*, allowing to regard the large-scale limit as a *coarse-graining* of the quantum network, where one discards an *ancillary system*. In order to do this, we find a closed-form expression for the *regularized* abelian walks and then we study the corresponding dispersion relation.
Since *non-abelian* QWS have been hardly studied, we will provide the first examples of non-abelian QWS satisfying the assumptions of *locality*, *homogeneity*, *isotropy* and *unitarity*, studying their dispersion relations as well. The regularization procedure is crucial in the non-abelian case, since it allows to define the *momentum* (otherwise it would not be possible) and hence to study the dynamical behaviour of non-abelian QWS as well.
The results obtained have relevance in the scope of studying the QWS' renormalized dynamics; furthermore, they trace a path for this model in comprehending whether the four assumptions are crucial in order to derive Dirac evolution and whether the renormalized QWS could exhibit a different kind of dynamics.

2 From [39]: "Emergence from events has an operational motivation in requiring that every physical quantity—including space-time—be defined through precise measurement procedures".

Part I

QUANTUM WALKS AND RENORMALIZATION

1

RANDOM WALKS

In this chapter a discrete model for the dynamical evolution of a system will be presented: the *random walk* (RW).
RWs are ubiquitous in science and play a role of great importance in many areas, such as physics, computer science, biology and quantitative finance.
We will introduce the concept of RW from a historical point of view, accounting then the recent applications and theoretical perspectives.

1.1 CLASSICAL RANDOM WALKS

Considering classical systems, *i.e.* moving particles, the idea of RW was introduced in science with the observations of the botanist Robert Brown in 1827, which are reported in [1].
Brown observed through a microscope the random motion of some pollen grains suspended in water and thus fed the long-standing debate about the existence of atoms. The phenomenon was regarded as an experimental evidence of the presence of fundamental "bricks" constituting matter and was named after its discoverer: *Brownian motion*.

In [2] Einstein theorized the emergence of Brownian motion as a statistical result due to collisions with molecules at a microscopic level; he obtained an equation for the evolution of the probability $p(x,t)$ of finding at x the macroscopic particle which undergoes the RW, recognising it as a *diffusion equation*.

Assigning a stochastic variable to the position of the particle, the general equation in 1D for the time evolution of the probability density function is

$$\partial_t p(x,t) = -\partial_x \left(\nu(x,t)p(x,t)\right) + \partial_x^2 \left(D(x,t)p(x,t)\right),$$

known as Fokker-Planck equation. D and ν are respectively the diffusion and the drift coefficients.
The processes described are time-continuous *Markov processes*, namely stochastic memoryless processes.

One can consider discrete processes as well, in both space and time. Therefore the particle will evolve, step by step, over neighbouring sites of a lattice, with some transition probability rules depending on the particular walk.

1.2 QUANTUM RANDOM WALKS

RWs were firstly extended to a quantum version in [3], where a one-dimensional discrete model is presented. Measurements of the z-component of a spin-$\frac{1}{2}$ particle are set up to discriminate the direction of motion at each step and the authors show that quantum interference effects the average path length, which can be much larger than the maximum allowed path in the corresponding classical RW. Subsequently, models for *quantum walks* (QWs) resorted to a unitary evolution for the system, accounting the evolution for both the position and the spin degrees of freedom; a unitary operator is used to describe the walk and the intrinsic degree of freedom is called *coin system* (see [4]).
Markov processes give interesting results studying even studying quantum dynamical systems, *e.g.* leading to generalized Schrödinger equations in the form of Fokker-Planck equations, *e.g.* in [41]. In [40], the authors derive a dispersive Schrödinger dfferential equation through a QW:

$$i\partial_t \psi(\mathbf{x},t) = \pm \left[\mathbf{v}\cdot\nabla + \frac{1}{2}\mathbf{D}\left(\nabla^T\nabla\right)\psi(\mathbf{x},t)\right]\psi(\mathbf{x},t)$$

(see next chapters for a discussion).

In the general case of a QW over a discrete lattice Λ with arbitrary internal degree of freedom, the coin system is represented by an Hilbert space $\mathscr{H}_x \cong \mathbb{C}^s$ (s integer) for each lattice site x: this is called a *cell structure* for the QW.

Definition 1.1 (Quantum walk)**.** *Let Λ be a lattice and N_x a finite subset of Λ. A* QW *over Λ with the* neighbourhood scheme N_x *is a unitary operator*

$$\begin{aligned} A \,:\, \mathscr{H}_x &\longrightarrow \bigoplus_{i\in N_x} \mathscr{H}_i \\ |\psi\rangle_x &\longmapsto A\,|\psi\rangle_x\,. \end{aligned}$$

The total Hilbert space of the system is the direct sum

$$\mathscr{H} = \bigoplus_{x\in\Lambda} \mathscr{H}_x$$

and by linearity of A one has

$$A\,|\psi\rangle = A\sum_{x\in\Lambda} |\psi\rangle_x = \sum_{x\in\Lambda}\sum_{y\in N_x} U_{xy}\,|\psi\rangle_y\,.$$

$\mathscr{H}$ is unitarily equivalent to $\ell^2(\Lambda)\otimes\mathbb{C}^s$, where $\ell^2(\Lambda)$ is the space of square summable complex functions defined on the lattice.

1.3 LATTICE MODELS FOR FREE QUANTUM FIELDS

In [42] Dirac pointed out the analogies between classical and quantum mechanics, proposing a method for discussing *trajectories* for the motion of a particle in quantum mechanics. This is a seminal Lagrangian method for reconciling a classical approach with quantum mechanics.
With [43], Feynman pushed forward Dirac's idea and gave a generalised procedure for a path-integral formulation of quantum mechanics.
These efforts arose also from the need of giving a relativistically invariant description of quantum system –which was totally lacking at the time– and were of crucial importance in order to give a mathematical formulation to *Quantum Field Theory* (QFT).

Serious mathematical problems initially plagued a perturbative approach to QFT, such as divergences in calculations, therefore a renormalization approach of the theory was needed and this procedure is now mathematically well founded (see *e.g.* in [44] by 't Hooft).
Unfortunately, QFT is still suffering from both conceptual and mathematical issues well explained in [45, 46].

On one hand, these difficulties arise from the fact that the renormalization schemes are of mathematical nature, therefore it is a quite abstract procedure, without a physical interpretation (Dirac himself, in [47], was critical about this solution); on the other hand, this approach is not suitable for the whole QFT: we know indeed that a quantum field theory of gravity is non-renormalizable.

For these reasons, lattice gauge theories have been taken into account as discrete approximations of QFT; *e.g.* Nakamura in [12] proposed a model for free quantum fields using non-standard analysis, giving Dirac equation in the *continuum* limit. In [14, 15], the authors reach the same limit considering a *quantum lattice gas*.
However, lattice theories were initially regarded as mere computational tools [13, 16]. In his pioneering work [17], Bialynicki-Birula firstly pointed out that Weyl dynamics on a lattice (in the *continuum* limit) necessarily follows from few assumptions.

The derivation of the renormalization schemes is carried on at the price of violating some basic physical tenets, *e.g.* Lorentz symmetry in *lattice-regularization*. In the QW's framework new phenomenology is theorized. Due to the discreteness of the model, all continuous symmetries of QFT are no longer valid, but spacetime and mechanics are emergent from an exact, fundamental mechanism. In this quantum network, isotropy is recovered through quantum superpositions of the different paths, recovering the usual dynamics of high-energy physics (Fermi scale).
Finally, as the authors notice in [40], it is interesting to point out that this model incorporates the main ingredients of the microscopic theories of gravity of Jacobson and Verlinde (see [48, 49]).

2

QUANTUM WALKS OVER CAYLEY GRAPHS

The formalism adopted resorts to a formulation based on *Group Theory* and *Graph Theory*. We will recall some basic definitions and results regarding these topics, pointing out their relevance for the scope of this work.

2.1 NOTIONS OF GROUP THEORY

The lattice sites are meant to be identified with elements of a group, therefore we recall the definition:

Definition 2.1 (Group). *Let* G *be a set and* $\bullet$ *a binary operation on* G*. The pair* $(G, \bullet)$ *is a* group *if it satisfies the following* group axioms*:*

- G *is closed under the binary operation,* i.e.

$$\begin{aligned} \bullet\ :\ G \times G &\longrightarrow\ G \\ (g_1, g_2) &\longmapsto\ g_1 \bullet g_2 = g_1 g_2 \end{aligned}$$

- $\forall g_1, g_2, g_3 \in G$, $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
- $\exists!\ e \in G$ *such that* $\forall g \in G$, $eg = ge = g$
- $\forall g \in G\ \exists\ g^{-1} \in G$ *(called* inverse *of* g*) such that* $g^{-1} g = g g^{-1} = e$

The result of the binary operation (or *group law*) depends in general on the order of composition of two elements of G. Groups for which this is not the case are called *abelian*.

Definition 2.2 (Generating set)**.** *Given a group* G*, a* generating set $S = S^+ \cup S^-$ *for the group (where* S^- *collects the inverses of* S^+*) is a subset such that every element of* G *can be written as a word* (i.e. *a combination under the group law) of finitely many elements of* S*.*
The elements of S^+ *are called* generators *of* G*. The cardinality of the smallest generating set of a group* G *is called* rank *of the group.*

From the group axioms follows that every element of G has unique inverse and this implies that S^+ and S^- are in one-to-one correspondence.

We now need a way to express the walk over element of a group.

Definition 2.3 (Irreducible representation)**.** *An* irreducible representation *(*irrep*)* Π *of a group* G *over a vector space* W *is a homomorphism from* G *to* $GL(W)$ *which has no invariant subspace but the trivial ones.*

A reducible representation for the QW would imply a splitting of the walk into direct sums of more elemental, irreducible walks.
In the following chapters we will consider unitary *completely reducible* representations, namely such that, $\forall g \in G$, $\Pi(g)$ is a direct sum of irreps which are unitary operators. As a useful results of Group Theory, we can mention that unitary irreducible representations of abelian groups are one-dimensional. Moreover, we will require Π injective, namely it is a *faithful representation*.

If a subset $H \subseteq G$ is itself a group under the binary operation defined on G, H is called a *subgroup* of G. In the present work we will be interested in partitions of a group induced by a subgroup: this motivates the following definitions.

Definition 2.4 (Left cosets)**.** *Let* G *be a group and* H *a subgroup of* G*. The* left cosets *of* H *in* G *are the equivalence classes defined by the equivalence relation*

$$g_1, g_2 \in G, \quad g_1 \sim g_2 \iff \exists\, h \in H \ : \ g_1 h = g_2$$

The left cosets of H in G are of the form $c_j H$, for some $c_j \in G$ (called *coset representatives*). Furthermore, by definition, every element of G belongs to one and only one left coset of H, therefore the left cosets are disjoint and induce a partition of G. Considering an alphabet J such that $\bigcup_{j \in J} c_j H = G$, the cardinality of J is called *index* of H in G.

Definition 2.5 (Virtually abelian group)**.** *Let* G *be a group and* H *a subgroup of* G*.* G *is said* virtually abelian *if* H *is abelian and has finite index in* G*.*

2.2 CAYLEY GRAPHS

We now provided some preliminary definitions in order to introduce the "ambient space" of the quantum walks considered: the *Cayley graph* of a group.

Definition 2.6 (Normal subgroup). *A subgroup* N *of a group* G *is called a* normal subgroup *if*

$$\forall n \in N \;\; \textit{and} \;\; \forall g \in G, \;\; g^{-1}ng \in N$$

A subgroup is normal if it is invariant under the operation just defined, which is called *conjugation* and defines an equivalence relation, the *conjugacy*.
If one aggregate equivalent element of a group, this preserves the group structure and gives rise to a:

Definition 2.7 (Quotient group). *Given a group* G *and a normal subgroup* N*, the* quotient group G/N *is the set of all the left cosets of* N *in* G *endowed with the group law of* G.

The next definitions are introductory to a very powerful tool, the *presentation of a group*, which is connected to Cayley graphs and turns out to be very useful in order to characterize and visualize groups in a variety of fashions.

Definition 2.8 (Free group). *Given a generating set* S*, the* free group F_S *generated by* S *is the group whose elements can be uniquely expressed words on* S.

Given an arbitrary group, the equality of two words on a generating set could follow from either the group axioms (particularly from the property of the inverses) or by virtue of some other relations which characterize the group itself, such as $a^m = e$ for some integer m (cyclic conditions) or $ab = ba$ (abelianity) (*e.g.* for a, b generators). Free groups lack of such relations and equalities between their elements follow solely from group axioms.

Definition 2.9 (Conjugate closure). *Given a group* F *and a set* R *of words, the* conjugate closure *of* R *in* F *is the group generated by the conjugates of* R*, namely the set:* $\{g^{-1}rg \mid g \in F, r \in R\}$.

Definition 2.10 (Presentation of a group). *Given a generating set* $S = S^+ \cup S^-$ *and a set* R *of words on* S*, let* F_S *be the free group on* S *and* N *the conjugate closure of* R *in* F_S*. Then*

$$G \sim \langle S^+ | R \rangle$$

is said a presentation *of the quotient group* $G = F_S/N$.

The words in R are called *relators*; since the conjugate closure of R is the smallest normal subgroup of F_S which contains R, one can take the quotient F_S/N and this assures that, in the resulting group G, the expressions in R will coincide with the identity. For the free group, in fact, R is trivial.

A given presentation completely specifies the properties of a group (modulo isomorphisms), but the *viceversa* is not true: there exist in general infinitely many possible presentations for a given group.

Definition 2.11 (Cayley graph)**.** *Let* G *be a group and* S^+ *a set of generators for* G*. Given a set* V *(whose elements are called* vertices*) and a set* E *whose elements are ordered pair of vertices (called* edges*), the* Cayley graph $\Gamma(G, S^+)$ *is the pair* (V, E) *constructed as follows:*

A. *a vertex is assigned to each* $g \in G$*;*

B. *a different color is assigned to each* $f \in S^+$*;*

C. *an edge is assigned to the pair* (g, fg) *for any* $g \in G$ *and* $f \in S^+$*.*

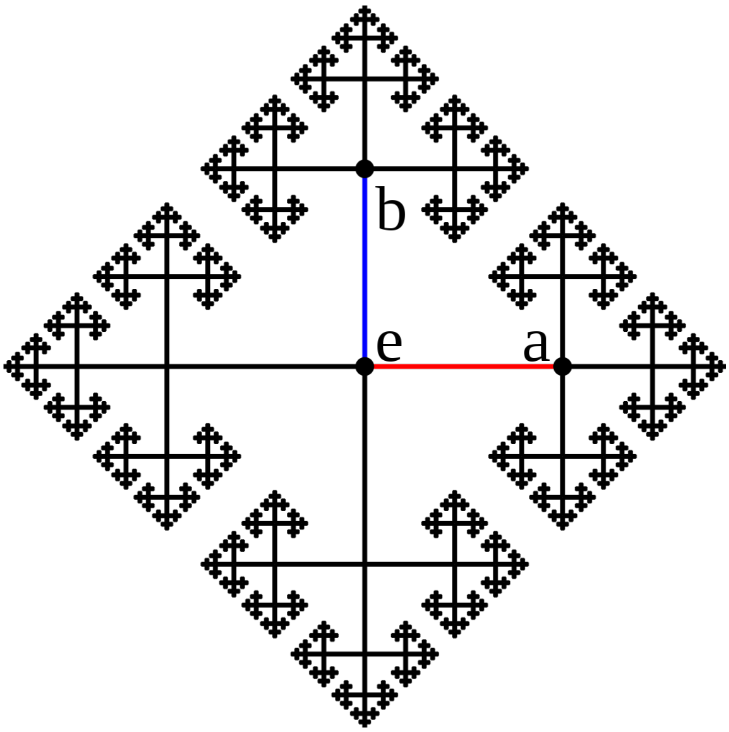

Figure 1: The Cayley graph of the free group.

One can see that different presentations of the same group give rise to different Cayley graphs, *e.g.*

$$\mathbb{Z} \sim \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$$

leads to a simple square lattice, while

$$\mathbb{Z} \sim \langle g_1, g_2, g_3 \mid g_i g_j g_i^{-1} g_j^{-1},\ g_2 (g_1 g_3)^{-1} \rangle$$

is associated with an hexagonal lattice for the group of integers.

A Cayley graph can be endowed with a metric:

Definition 2.12 (Word metric)**.** *For any* g *element of* G*, its* word norm $|g|$ *with respect to a generating set* S *is the number of letter of the shortest word over* S *which is equal to* g*.*
Given $g_1, g_2 \in G$ *such that* $g_1 w = g_2$*, the distance* $d(g_1, g_2)$ *in the word metric is* $|w|$*.*

A lattice together with the word metric (also called *counting metric*) becomes a metric space. In the following, we will consider mapping from Cayley graph to Euclidean spaces and it is convenient to define a correspondence which allows to compare metrics between them. This is provided by a quasi-isometric embedding.

Definition 2.13 (Quasi-isometric embedding)**.** *Given two metric spaces* (M_1, d_1) *and* (M_2, d_2)*, a function*

$$f \,:\, (M_1, d_1) \longrightarrow (M_2, d_2)$$

is a quasi-isometric embedding *if there exist some constants* $A \geqslant 1$ *and* $B, C \geqslant 0$ *such that*

- $\forall u, v \in M_1,\ \frac{1}{A} d_1(u, v) - B \leqslant d_2(f(u), f(v)) \leqslant A d_1(u, v) + B$
- $\forall w \in M_2\ \ \exists\, z \in M_2$ *such that* $d_2(w, f(z)) \leqslant C$

The counting metric is not equivalent to the Euclidean metric due to the so called Weyl tile argument (see [50]). The existence of a quasi-isometric embedding between *e.g.* a simple square d-dimensional lattice and $\mathbb{R}^d$ means, intuitively, that the two metrics are equivalent apart from the constant bound $\sqrt{d}$.

If there exist a quasi-isometric embedding between two Cayley graphs, the underlying groups are said *quasi-isometric*.

The last important result in the scope of this work is the following:

Theorem 2.1 (Quasi-isometric rigidity of $\mathbb{Z}^d$)**.** *If a finitely generated group is quasi-isometric to $\mathbb{Z}^d$, then it has a finite index subgroup isomorphic to $\mathbb{Z}^d$.*

Remark 2.1. *Since $\mathbb{Z}^d$ is quasi-isometric to $\mathbb{R}^d$ and quasi-isometry is an equivalence relation, we can state that if a finitely generated group G is quasi-isometric to $\mathbb{R}^d$, then G is virtually abelian.*

With regard to section 1.2, we conclude this chapter pointing out that in the following we will consider QWs defined on $\ell^2(G) \otimes \mathbb{C}^s$, for some group G whose Cayley graph satisfies the physical assumptions exposed in the next chapter.

3

WEYL AND DIRAC EQUATIONS FROM QWs

We will enunciate the assumptions of the model presented, together with the general framework; then the derivations of Weyl and Dirac QWs will be retraced in their main points.

3.1 ASSUMPTIONS AND FORMALISM

The scope of [40] is to derive the standard description of free quantum fields as emergent from a QW on a lattice, whose sites are a denumerable set of identical quantum systems in mutual interaction.
As already discussed, QWs over Cayley graphs are those under study; in order to recover the usual Euclidean space and metric in the limit of large scales, the authors require that the Cayley graph is quasi-isometrically embeddable in $\mathbb{R}^d$, namely the underlying group is virtually abelian.

The main physical assumptions are:

1. *Locality*: every site interacts with a finite number of nearest neighbour sites;
2. *Homogeneity*: universality of the physical law;
3. *Isotropy*: there is no favoured direction of interaction;
4. *Unitarity*: the evolution operator is a unitary operator.

These can be easily formalized giving a group-theoretical description of the walk over a Cayley graph. Intuitively, each site is identified with an element g of a group G and is linked by generators to a set of other sites, which constitute the nearest neighbours in mutual interaction with g.
Since the lattice is supposed to be an infinite and denumerable set and it will be identified with G, the group under consideration will be an infinite group endowed with discrete topology.

The *locality* assumption implies that the cardinality of the generating set S if finite, *i.e.* G is a finitely generated group.
With regards to *homogeneity*, a distinction can be done between a *weak* concept of *homogeneity* and a *strong* one. The *weak* assumption states that the cardinality of S must not depend on the site (which is trivially granted by the definition itself of Cayley graph). On the other hand, in [40] a *stronger* assumption of *homogeneity* is required: the walk must be translationally invariant, namely the evolution operator commutes with the shift operator $\Sigma(f)$ on the lattice site—defined as $\Sigma(f)\,|\psi\rangle_x = |\psi\rangle_{fx}$ for $f \in S$—namely

$$A\Sigma(f)\,|\psi\rangle_x = \sum_{x' \in N_{fx}} |\psi\rangle_{x'} = \Sigma(f)A\,|\psi\rangle_x = \sum_{x' \in N_x} |\psi\rangle_{fx'}\,.$$

This entails Cayley graphs $\Gamma(G, S)$ with G abelian. This abelianity assumption is not the most general, since it rules out virtually abelian groups, however it implies $G \cong \mathbb{Z}^d$.[1]
The evolution of the walk is represented by an evolution operator A, whose general expression is

$$A = \sum_f T_f \otimes A_f, \tag{1}$$

acting on the Hilbert space $\ell^2(G) \otimes \mathbb{C}^s$ ($T_g = \sum_{g' \in G} |g \cdot g'\rangle \langle g'|$ being a unitary, faithful and completely reducible representation of the group G).
The *isotropy* assumption can be formalized into the minimal requirement that there exists a faithful unitary representation U_c of some group transitive over S^+ such that the invariance

$$A = \sum_{f \in S} T_f \otimes A_f = \sum_{f \in S} T_{cf} \otimes U_c A_f U_c^\dagger \tag{2}$$

1 Anyhow, it is interesting to notice that the *homogeneity* assumption is a purely topological property, not a metrical one.

under the action of this group holds[2].
The hypothesis of *unitarity* is the most natural one, since it is a standard requirement that time evolution must preserve norms.

Imposing unitarity on A, one can find the transition matrices $\{A_f\}_{f\in S}$ for the coin system of the walk (such a derivation makes use of the *isotropy* assumption as well).
Accordingly, the unitarity conditions are

$$AA^\dagger = A^\dagger A = T_e \otimes \mathbb{1}_s,$$

which imply

$$\sum_{ff'^{-1}=g} A_f A^\dagger_{f'} = \sum_{f^{-1}f'=g} A^\dagger_f A_{f'} = 0 \quad \text{(with } g \neq e) \tag{3}$$

and the normalization condition

$$\sum_{f\in S} A_f A^\dagger_f = \sum_{f\in S} A^\dagger_f A_f = \mathbb{1}_s. \tag{4}$$

Finally, it can be shown that as necessary condition for *isotropy* one can always assume, modulo a local unitary,

$$\sum_{f\in S} A_f = \mathbb{1}_s. \tag{5}$$

Assuming therefore $G = \mathbb{Z}^d$, one can see that the states of the form

$$|\mathbf{k}\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}\in G} e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}\rangle \tag{6}$$

are the joint eigenvectors of the translations T_g, with eigenvalues $e^{i\mathbf{k}\cdot\mathbf{g}}$:

$$T_g |\mathbf{k}\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}\in G} e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}+\mathbf{g}\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}'\in G} e^{-i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{g})} |\mathbf{x}'\rangle$$

2 We point out that the *isotropy* assumption allows to select *one* particular walk (or a restricted class of walks sharing common properties) among an infinite admissible family.

$$= \frac{e^{i\mathbf{k}\cdot\mathbf{g}}}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}'\in G} e^{-i\mathbf{k}\cdot\mathbf{x}'} \left|\mathbf{x}'\right\rangle \equiv e^{i\mathbf{k}\cdot\mathbf{g}} \left|\mathbf{k}\right\rangle,$$

where abelian notation is used. In fact, $e^{i\mathbf{k}\cdot\mathbf{g}}$ is simply a one-dimensional unitary irrep of an abelian group isomorphic to $\mathbb{Z}^d$, while the class of states (6) are just the wavefunctions of the field in the *k-space*.
The first Brillouin zone is defined by the following constraint:

$$\mathscr{B} = \left\{\mathbf{k}\middle| -\pi \leqslant \mathbf{k}\cdot\tilde{\mathbf{f}}_j \leqslant \pi,\ j = 1,\ldots,d\right\},$$

where $\left\{\tilde{\mathbf{f}}_j\right\}_{j=1}^{d}$ is the dual basis of $\mathbb{Z}^{d^*}$, for any choice of independent vectors $\{\mathbf{f}_j\}_{j=1}^{d}$ in the direct space.
We can therefore use the eigenvectors $|\mathbf{k}\rangle$ in order to diagonalize the T_f and, since the automaton is invariant under translations, equation (1) thus reads

$$\begin{aligned} A &= \sum_f \int_{\mathscr{B}} d\mathbf{k}\, |\mathbf{k}\rangle \langle\mathbf{k}|\, e^{i\mathbf{k}\cdot\mathbf{f}} \otimes A_f \\ &= \int_{\mathscr{B}} d\mathbf{k}\, |\mathbf{k}\rangle \langle\mathbf{k}| \otimes \left(\sum_f e^{i\mathbf{k}\cdot\mathbf{f}} A_f\right) =: \int_{\mathscr{B}} d\mathbf{k}\, |\mathbf{k}\rangle \langle\mathbf{k}| \otimes A_{\mathbf{k}}. \end{aligned} \tag{7}$$

Through this representation of G, it is convenient to interpret $\mathbf{k}\cdot\tilde{\mathbf{f}}_j$ as the momentum components, likewise to the standard procedure in solid state physics (specifically studying crystal structures).

The field undergoes a discrete-time evolution governed by the unitary operator A: the n-th discrete-time step is given by

$$|\psi(n)\rangle = (A)^n |\psi(0)\rangle.$$

In the momentum representation, the eigenvalues of (7) will be of the form $e^{i\omega_j(\mathbf{k})}$, where $\omega_j(\mathbf{k})$ can be interpreted as a dispersion relation, *i.e.* an energy versus momentum relation[3].

3 As mentioned in section 1.2, the QW's large scale limit gives rise to a dispersive Schrödinger equation, whose drift coefficient is $\frac{\partial\omega}{\partial k_j}$ (as a *group velocity*) and the diffusion one is the matrix $\frac{\partial^2\omega}{\partial k_i \partial k_j}$; interpreting ω as an energy is a usual analogy in physics, whereas in the present context this interpretation of the dispersion relation needs a fully interacting theory, where energy can be exchanged between systems. However, the analogy here considered for free fields can be well founded *a posteriori* at least in the relativistic limit, in view of the emergence of the usual field equations. In this framework, the Planck length l_P, time t_P and mass m_P conceptually represent

In the cases $s = 2$ and $d = 1, 2, 3$ for a massless field, the unique admissible (with respect to the initial assumptions) Cayley graphs of $\mathbb{Z}^d$ are found to be those whose vertices are the the integers lattice for $d = 1$, the simple square lattice for $d = 2$ and the BCC lattice (*body-centered cubic*) for $d = 3$. The assumption of unitarity is crucial in order to discriminate between the different topologically inequivalent graphs and select the unique valid set of transition matrices (modulo some symmetries). It is interesting to notice that in $d = 2$ can be shown that if one takes $e \in S$, A_e must be vanishing.
Since in this work we will take into account Cayley graphs quasi-isometrically embeddable in $\mathbb{R}^2$, we show the results for the 1D and the 2D walks.

3.2 THE WEYL QW

In both 1D and 2D cases, the eigenvalues of (7) are of the form $\omega_{\pm}(\mathbf{k})$ and give in the relativistic limit (small momenta) the dispersion relation

$$\omega_{\pm}(\mathbf{k}) \xrightarrow[|\mathbf{k}| \ll 1]{} \pm|\mathbf{k}|.$$

The one-dimensional case is trivial from this point of view, since the transition matrices are

$$A_{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{-} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\omega_{\pm}(k) = \pm k$ already holds at all scales.
In the two-dimensional case, instead, the transition matrices are

$$\begin{aligned} A_{+x} &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad A_{+y} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \\ A_{-x} &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_{-y} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{8}$$

and the dispersion relation is given by

$$\omega_{\pm}(k_x, k_y) = \pm \arccos\left[\frac{1}{2}(\cos k_x + \cos k_y)\right]. \tag{9}$$

just digital-analog conversion factors in order to recover physical quantities at the usual Fermi scale of high-energy physics ($\hbar = m_P l_P^2 t_P^{-1}$).

However, both the 1D and the 2D case exactly approximate the usual dispersion relation.

Moreover, an asymptotic approach for the time evolution is performed defining an *interpolating Hamiltonian* as

$$e^{-iH_I(\mathbf{k})} := A_{\mathbf{k}}$$

and taking its *finite-difference* counterpart, namely $\mathfrak{Im}\left(A_{\mathbf{k}}\right) = \sin H_I\left(\mathbf{k}\right)$. Inverting this last expression and expanding $H_I\left(\mathbf{k}\right)$ to the first order in $\mathbf{k}$, finally

$$i\partial_t \left|\psi\left(\mathbf{k},t\right)\right\rangle = H_I\left(\mathbf{k}\right)\left|\psi\left(\mathbf{k},t\right)\right\rangle \simeq \frac{1}{\sqrt{d}}\boldsymbol{\sigma}\cdot\mathbf{k}\left|\psi\left(\mathbf{k},t\right)\right\rangle$$

holds and the Weyl equation is recovered in the asymptotic limit.

3.3 THE DIRAC QW

With regard to the massive case, the identical transition T_e is introduced.
In 1D a QW for $s=2$ can be derived with the same methods presented *e.g.* in appendix A.1, since a Dirac spinor is two-dimentional in $d=1$.
The corresponding transition matrices turn out to be:

$$A_+ = \begin{pmatrix} \nu & 0 \\ 0 & 0 \end{pmatrix}, \quad A_- = \begin{pmatrix} \nu & 0 \\ 0 & 0 \end{pmatrix}, \quad A_e = \begin{pmatrix} 0 & im \\ im & 0 \end{pmatrix}.$$

where m,ν are reals and $m^2+\nu^2=1$. The parameter m is the mass of the walk.
In 2D it is needed $s=4$, thus two Weyl walks are coupled in a suitable way and the mass term is alike introduced.
It is then shown that the only possible local fashion to couple two walks is

$$A^D = \begin{pmatrix} \nu A_{\mathbf{k}} & im\mathbb{1} \\ im\mathbb{1} & \nu A_{\mathbf{k}}^\dagger \end{pmatrix} \equiv \sum_f e^{i\mathbf{k}\cdot\mathbf{f}} \begin{pmatrix} \nu A_f & 0 \\ 0 & \nu A_f^\dagger \end{pmatrix} + im\left(\sigma_x \otimes \mathbb{1}\right), \tag{10}$$

where, again, $m^2 + \nu^2 = 1$.
The phases of the eigenvalues give the dispersion relation

$$\omega_{\pm}\left(k_x, k_y, \nu\right) = \pm \arccos\left[\frac{\nu}{2}\left(\cos k_x + \cos k_y\right)\right]. \tag{11}$$

in the 2D case (the 1D is obtained identifying $k_x = k_y$).
The relativistic limit (small momentum and mass) gives

$$\omega_{\pm}\left(\mathbf{k}, \nu\right) \xrightarrow[|\mathbf{k}|\ll 1]{} \pm\sqrt{k_1^2 + k_2^2 + m^2}.$$

The approximate Dirac equation is restored with the same asymptotic approach of Weyl approximate equation.
The *interpolating Hamiltonian* gives

$$i\partial_t \left|\psi\left(\mathbf{k}, \nu, t\right)\right\rangle \simeq \left(\frac{\nu}{\sqrt{d}}\boldsymbol{\alpha}\cdot\mathbf{k} + m\beta\right)\left|\psi\left(\mathbf{k}, t\right)\right\rangle := H_D\left(\mathbf{k}, \nu\right)\left|\psi\left(\mathbf{k}, t\right)\right\rangle,$$

where $\nu \xrightarrow[m\ll 1]{} 1$ and so $H_D\left(\mathbf{k}, \nu\right)$ tends to the Dirac Hamiltonian and

$$\begin{aligned}\beta &= \gamma_0 \equiv \mathbb{1}\otimes\sigma_x,\\ \alpha_i &= \gamma_0\gamma_i,\\ \gamma_i &\equiv \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0\end{pmatrix},\end{aligned}$$

in *Weyl* (or *chiral*) representation.

4

RENORMALIZATION FRAMEWORK

4.1 MOTIVATIONS

As it has been illustrated in the previous chapters, from the physical point of view it is interesting to study the behaviour of a QW in the neighbourhood of local minima of the dispersion relation versus the momenta[1]. In the works presented, this approach allowed to investigate the limit of the QW for small momenta and to find a differential equation, linear in $\mathbf{k}$, describing the large scale dynamics of the walk.

This procedure is exactly performed in the momentum space, but it does not carry any piece of information about the positions (or *direct*) space, namely it does not allow to rewrite a large *space* scale version of the evolution operator of the QW. This is a first, straightforward motivation of the new approach to the asymptotic limit that is provided in the present work.
Moreover, the formalism presented so far exploits assuming the hypothesis of abelianity of the group underlying the walk. This is a radical assumption, but it is crucial in order to be able to define a momentum space of the QW and thus study its dynamical behaviour at any scale.
Nevertheless, the more compelling and fundamental request made on the group pertains its Cayley graph and is the quasi-isometric embeddability in $\mathbb{R}^d$, which traces a direction to the purpose of recovering some direct space as emergent.

1 Following the analogy discussed in chapter 3, we can say that, intuitively, a minimum of energy corresponds to a particle state.

However, from section 2.2 we know that abelian groups are not the only class of groups holding this feature, which is granted in general by virtually abelian groups (of which abelian group are a special case).
Accordingly, there is no profound reason to rule out virtually abelian groups *a priori*, but, at the same time, taking into account non-abelian groups gives rise to difficulties in providing a natural definition of a mathematical object to be interpreted as the momentum.
Abelian groups, in fact, admit one-dimensional unitary irreducible representations, which are then parametrized by complex unitary numbers whose phases are interpretable as momenta. However, unitary irreducible representations of non-abelian groups in general are not one-dimensional. Even if one found a non-abelian QW, yet there would not be a straightforward notion of momentum, so that a differential equation in the momentum space would be hardly derived. Accordingly the second motivation to study a new asymptotic approach is finding a procedure to induce a dispersion relation versus some notion of momentum for a non-abelian walk.

4.2 BASIC IDEAS: REGULARIZATION AND RENORMALIZATION

The idea of regularizing the description of a lattice in order to study its statistical behaviour at a larger scale resorts to a procedure first proposed in [51]. In this work, the interacting sites of the Ising model are regularly grouped into larger cells which are then treated as single sites of a new, coarse-grained lattice. Averaging the behaviour of the resulting blocks, one is then able to make a statistical description of the lattice; this procedure is know as *block-spin renormalization*.

In the scope of the present work, QWs over virtually abelian groups are taken into account. Exploiting the definition of virtual abelianity, one can attempt a sort of "coarse-graining" of the Cayley graph considered, regularizing the lattice and resorting to an abelian version of it.
The basic idea is to perform a tiling of the lattice and this is achieved partitioning the group G under consideration into left cosets of an abelian subgroup H. The vertices of the Cayley graph of G are thus grouped into elemental tiles that tessellate the embedding space and become the nodes of a new, coarse-grained graph.
This procedure consists in finding a regularized expression of the walk in terms of the generators of H and assigning the behaviour of the single sites of the orig-

inal group G to additional degrees of freedom, which are eventually discarded in order to study the renormalized action of the walk.

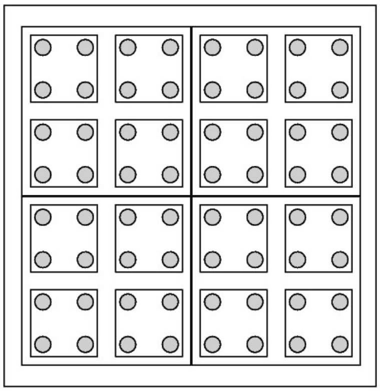

Figure 2: An example of block-site regularization.

4.3 RENORMALIZING THE QWS

Firstly we will provide a formal definition of the intuitive notion of "tiling of a lattice".

Definition 4.1 (Regular tiling). *Let* G *be a virtually abelian group and let* H *be an abelian subgroup of* G *of finite index. If* G *is a finitely generated infinite group and* $H \cong \mathbb{Z}^d$*, we say that a left cosets partition*

$$\bigcup_{j=1}^{l} c_j H = G \quad (\textit{with } c_1 = e)$$

is a regular tiling (of order l*) of the Cayley graph of* G*.*

It is implicit in the previous definition that the Cayley graph of G is quasi-isometrically embeddable in $\mathbb{R}^d$, since otherwise the subgroup $H \cong \mathbb{Z}^d$ would be of infinite index.
In general the cosets partition is not unique, namely the cosets representatives are not uniquely determined by H; nevertheless, only the proprieties of H in G are relevant in order to carry on the regularization of a QW and not the choice of the c_j, which is arbitrary in this context.

Remark 4.1. *Being* H *a group, the sublattice defined by the first coset is the Cayley graph corresponding to some presentation of* $\mathbb{Z}^d$*.*

Since the cosets are mutually disjoint by definition (they define an equivalence class), the lattices defined by all the cosets $c_j H$ *are mutually disjoint sublattices of the Cayley graph of* G*; moreover, since* $\bigcup_{j=1}^{l} c_j H = G$*, the union of these sublattices fulfils the whole Cayley graph.*
Chosen a regular tiling, the cosets representatives c_j *are elements of* G *whose action on the subgroup* H *allows to move between the sublattices and one can consider the collection of the cosets representatives* $\{c_1, c_2, \dots, c_l\}$ *as the elemental tile of this regular tessellation.*

Remark 4.2. *In case of* G *isomorphic to* $\mathbb{Z}^d$*, any subgroup* $H \cong \mathbb{Z}^d$ *defines a regular tiling of* G.
A suitable kind of tessellation for the purposes of the present work will be taken into account when studying QWs *over purely abelian groups.*

In view of the scope of the regularization procedure, it is worth noting the following important property of a coset partition: from the disjointness of the cosets it follows that every $g \in G$ admits a unique decomposition in terms of c_j and $\mathbf{x} \in H$. This means that each element of G is in one-to-one correspondence with an element of the form $(\mathbf{x}, c_j)$. QW's evolution in the direct space is represented on $\mathscr{H} = \ell^2(G)$: it is then useful to try to split this space exploiting the correspondence just showed.
Indeed, since all infinite-dimensional separable Hilbert spaces are isometrically isomorphic, one can find an isometric mapping between $\mathscr{H}$ and $\mathscr{K} = \ell^2(H) \otimes \mathbb{C}^l$ (for any given integer l).
Given an orthonormal basis $\{|j\rangle\}_{j=0}^{l-1}$ for $\mathbb{C}^l$, let's define the isometry

$$\begin{aligned} U_H \, : \mathscr{H} &\longrightarrow \mathscr{K}, \\ |c_j \mathbf{x}\rangle &\longmapsto |\mathbf{x}\rangle \otimes |j\rangle, \end{aligned}$$

represented by the operator

$$U_H := \sum_j \sum_{\mathbf{x} \in H} |\mathbf{x}\rangle |j\rangle \langle c_j \mathbf{x}| \, .$$

It is immediate to check that U_H is an isometry:

$$\begin{aligned} U_H^\dagger U_H &= \sum_{j,j'} \sum_{\mathbf{x},\mathbf{x}' \in H} |c_j \mathbf{x}\rangle \langle c_{j'} \mathbf{x}'| \langle \mathbf{x} | \, \mathbf{x}'\rangle \langle j | \, j'\rangle \\ &= \sum_j \sum_{\mathbf{x} \in H} |c_j \mathbf{x}\rangle \langle c_j \mathbf{x}| = T_e^{\mathscr{H}} \equiv \mathbb{1}_{\mathscr{H}}; \end{aligned}$$

furthermore, due to the uniqueness of the decomposition of G previously defined, it is also a unitary mapping:

$$
\begin{aligned}
U_H U_H^\dagger &= \sum_{j,j'} \sum_{\mathbf{x},\mathbf{x}'\in H} |\mathbf{x}\rangle \langle \mathbf{x}'| \otimes |j\rangle \langle j'| \underbrace{\langle c_j\mathbf{x}|\, c_{j'}\mathbf{x}'\rangle}_{\delta_{j,j'}\delta_{\mathbf{x},\mathbf{x}'}} \\
&\equiv \sum_{\mathbf{x}\in H} |\mathbf{x}\rangle \langle \mathbf{x}| \otimes \sum_j |j\rangle \langle j| = T_e^{\mathscr{H}} \equiv \mathbb{1}_{\mathscr{H}}.
\end{aligned}
$$

We can exploit the fact that the unitary irreducible representations of an abelian group are one-dimensional and define, even in non-abelian cases, the following classes of states of $\mathscr{H}$:

$$
|\mathbf{k}\rangle_j := \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}\in H} e^{-i\mathbf{k}\cdot\mathbf{x}} |c_j\mathbf{x}\rangle ,
$$

which are mapped to

$$
\begin{aligned}
U_H |\mathbf{k}\rangle_j &= \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{j'} \sum_{\mathbf{x},\mathbf{x}'\in H} e^{-i\mathbf{k}\cdot\mathbf{x}} \langle c_{j'}\mathbf{x}'|\, c_j\mathbf{x}\rangle |\mathbf{x}'\rangle |j'\rangle \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}\in H} e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}\rangle |j\rangle \equiv |\mathbf{k}\rangle_H |j\rangle .
\end{aligned}
$$

As already discussed in section 4.1, if G is non-abelian it is not possible to diagonalize (and thus regularize) the translations on G directly over the $|\mathbf{k}\rangle_j$, because

$$
\begin{aligned}
T_f |\mathbf{k}\rangle_j &= \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}\in H} e^{-i\mathbf{k}\cdot\mathbf{x}} |fc_j\mathbf{x}\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\mathbf{x}'\in H} e^{-i\mathbf{k}\cdot\left(c_j^{-1}f^{-1}c_{j'(f)}\mathbf{x}'\right)} |c_{j'(f)}\mathbf{x}'\rangle \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-i\mathbf{k}\cdot\left(c_j^{-1}f^{-1}c_{j'(f)}\right)} \sum_{\mathbf{x}'\in H} e^{-i\mathbf{k}\cdot\mathbf{x}'} |c_{j'(f)}\mathbf{x}'\rangle \equiv e^{-i\mathbf{k}\cdot\left(c_j^{-1}f^{-1}c_{j'(f)}\right)} |\mathbf{k}\rangle_{j'(f)} ,
\end{aligned}
$$

(since in the change of variable $c_j^{-1}f^{-1}c_{j'(f)}$ is equal to $\mathbf{x}\mathbf{x}'^{-1} \in H$ by definition) and in general $j'(f) \neq j$.

Remark 4.3. *Note that, if the subgroup* H *is normal, the* $|\mathbf{k}\rangle_j$ *are the invariant spaces of* T_h*, the generators of the translations of* H*:*

$$
\forall \;\; h \in S_H \;\; \exists \;\; \tilde{x} \in H \;\; : \;\; hc_j = c_j\tilde{x},
$$

and one has

$$T_h |\mathbf{k}\rangle_j = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{x \in H} e^{-i\mathbf{k}\cdot\mathbf{x}} \left|hc_j\mathbf{x}\right\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{x' \in H} e^{-i\mathbf{k}\cdot(\mathbf{x}'-\tilde{\mathbf{x}})} \left|c_j\mathbf{x}'\right\rangle \equiv e^{i\mathbf{k}\cdot\tilde{\mathbf{x}}} |\mathbf{k}\rangle_j \,.$$

In $\mathscr{K}$ *we have*

$$\begin{aligned}\left(U_H T_f U_H^\dagger\right) \cdot |\mathbf{k}\rangle_H |j\rangle &= \left(U_H T_f U_H^\dagger\right) \cdot U_H |\mathbf{k}\rangle_j \\ &= U_H \cdot T_f |\mathbf{k}\rangle_j \equiv e^{-i\mathbf{k}\cdot\left(c_j^{-1} f^{-1} c_{j'(f)}\right)} |\mathbf{k}\rangle_H \left|j'(f)\right\rangle,\end{aligned}$$

namely $|\mathbf{k}\rangle_H$ *is the invariant space of the* $U_H T_f U_H^\dagger := \tilde{T}_f$*.*

The regularized generators $\tilde{T}_f$ map $\ell^2(H)$ into itself (up to a phase factor dependent on the generator f) and possibly change the new "internal" degree of freedom on $\mathbb{C}^l$. Then this mapping allows to diagonalize the generators of G over the momentum space of $\mathscr{K}$ exploiting their action on the additional degrees of freedom.
The regularized evolution operator will be given by

$$\mathcal{R}[A] := (U_H \otimes \mathbb{1}) A (U_H \otimes \mathbb{1})^\dagger = \sum_f \tilde{T}_f \otimes A_f. \quad (12)$$

4.4 THEORETICAL APPLICATIONS

As already pointed out, this procedure works for virtually abelian groups in general and then in particular for abelian groups.

The cases of walks over $G = \mathbb{Z}^d$ have been extensively studied and the dispersion relations for the unique admissible Weyl and Dirac QWs (for $d = 1, 2, 3$) have been found as discussed.
The renormalization procedure allows to re-express these walks in a *regularized* form after having fixed a regular tessellation of the lattices for $d = 1, 2$; we will then iterate the regularization scheme to the scale desired, reaching a new expression for the abelian walks.
After having regularized the abelian walks, their action on $\ell^2(H)$ is obtained marginalizing on $\mathbb{C}^l$: one applies the regularized walk on the density operators on $\ell^2(H) \otimes \mathbb{C}^l$ and then traces out the additional degree of freedom.
Finally, two points will be discussed:

- One could ask if and how the dispersion relation of the coarse-grained walks changes.
- One can apply to some particular states the regularized walk.

On the other hand, two elementary non-abelian QWs in 2D will be derived and this procedure will be applied to them, in order to recover a physical momentum and study their dispersion relations.

Part II

APPLICATIONS OF THE RENORMALIZATION FRAMEWORK

5

RENORMALIZATION OF QWs OVER ABELIAN GROUPS

In the following, we shall make use of most of the results shown in chapter 3.

5.1 ONE-DIMENSIONAL WEYL QW

In the 1D case we have $G = \mathbb{Z}$: the group presentation is trivial (one generator, no relators), we can use the additive notation for the group composition and the only generator will be denoted by 1.
A regular tiling of order l is given by

$$\mathbb{Z} = \bigcup_{j=0}^{l-1} (c_j + H) \,,$$

where the cosets representatives are $c_j = j$ for $j = 0, \ldots, l)$, $H = l\mathbb{Z} \cong \mathbb{Z}$ and its generator is l[1].
Let's define the invariant spaces under $T_{\pm l}$:

$$|k\rangle_j := \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{x \in H} e^{-ixk} |x + j\rangle \,.$$

1 Technically, $(l\mathbb{Z}, +)$ is an *ideal* of the ring $(\mathbb{Z}, +, \cdot)$; however it has a group structure isomorphic to $\mathbb{Z}$ and we will treat it as a group. Accordingly, in the following we shall make use of the notation lx in order to indicate the generator of $l\mathbb{Z}$ expressed in terms of the generator of $\mathbb{Z}$: this notation stands for "l-th power of x under the additive group law".

Let's now evaluate the regularized generators of the translations in $\ell^2(H) \otimes \mathbb{C}^l$, namely $\tilde{T}_{\pm 1} = U_H T_{\pm 1} U_H^\dagger$; by definition of U_H (provided a convenient scaling of the element of H), one has:

$$\begin{aligned}\tilde{T}_{+1} &= \sum_{j,j'=0}^{l-1} \sum_{x,x' \in \mathbb{Z}} |x\rangle \langle x'| \otimes |j\rangle \langle j'| \underbrace{\langle lx + j| T_{+1} |lx' + j'\rangle}_{=\delta_{lx+j,lx'+j'+1}} \\ &= \sum_x |x+1\rangle \langle x| \otimes |0\rangle \langle l-1| + \sum_x |x\rangle \langle x| \otimes \sum_{j=1}^{l-1} |j+1\rangle \langle j|\end{aligned} \tag{13}$$

and

$$\begin{aligned}\tilde{T}_{-1} &= \sum_{j,j'=0}^{l-1} \sum_{x,x' \in \mathbb{Z}} |x\rangle \langle x'| \otimes |j\rangle \langle j'| \underbrace{\langle lx + j| T_{-1} |lx' + j'\rangle}_{=\delta_{lx+j,lx'+j'-1}} \\ &= \sum_x |x-1\rangle \langle x| \otimes |l-1\rangle \langle 0| + \sum_x |x\rangle \langle x| \otimes \sum_{j=1}^{l-1} |j-1\rangle \langle j|,\end{aligned} \tag{14}$$

namely the $\tilde{T}_{\pm 1}$ perform a periodic shift among the different cosets of H in G (*i.e.* on the additional degree of freedom of $\mathbb{C}^l$). Furthermore, every matrix element of the shift is associated either with a translation or with the identical transition on the "coarse-grained space" $\ell^2(H)$; one can simultaneously diagonalize T^H_{+1} and T^H_e over their joint eigenvectors in momentum space:

$$\begin{aligned}T^H_e &\equiv \sum_x |x\rangle \langle x| = \int_{\mathscr{B}} dk\, |k\rangle \langle k|, \\ T^H_\pm &\equiv \sum_x |x \pm 1\rangle \langle x| = \int_{\mathscr{B}} dk\, |k\rangle \langle k|\, e^{\pm ik}.\end{aligned}$$

Accordingly, one can easily express regularized walk on $\ell^2(H) \otimes \mathbb{C}^l$ trough the $l \times l$ *shift matrices*

$$Z_l^+ := \begin{pmatrix} 0 & 0 & \cdots & 0 & e^{ik} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad Z_l^- := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ e^{-ik} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

These are unitary matrices for which the identity $Z_l^+ = \left(Z_l^-\right)^\dagger$ holds (they are each other's inverse). Moreover, one can easily show that the eigenvalues of $Z_l^\pm$ are just $e^{\pm i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}$ $(j = 0, \ldots, l-1)$.

Equation (12) reads

$$\begin{aligned}
\mathcal{R}[A] &= \int_{\mathscr{B}} dk \,|k\rangle\langle k| \otimes Z_l^+ \otimes A_{+1} + \int_{\mathscr{B}} dk\,|k\rangle\langle k| \otimes Z_l^- \otimes A_{-1} \\
&= \int_{\mathscr{B}} dk\,|k\rangle\langle k| \otimes \left(Z_l^+ \otimes A_{+1} + Z_l^- \otimes A_{-1}\right) =: \int_{\mathscr{B}} dk\,|k\rangle\langle k| \otimes \mathcal{R}[A]_k\,,
\end{aligned}$$

and the new dispersion relation is obtained diagonalizing $Z_l^\pm$:

$$\begin{aligned}
\mathcal{R}[A]_k &= \left(\sum_j e^{+i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}|j\rangle\langle j|\right) \otimes A_{+1} + \left(\sum_j e^{-i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}|j\rangle\langle j|\right) \otimes A_{-1} = \\
&= \sum_j |j\rangle\langle j| \otimes \underbrace{\left(e^{+i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}A_{+1} + e^{-i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}A_{-1}\right)}_{:=\mathcal{W}(k,j)}. \qquad (15)
\end{aligned}$$

The diagonal blocks in (15) represent just l Weyl walks with shifted and scaled momenta and it's straightforward to find the dispersion relations

$$\omega_j(k) = \pm\left(\frac{k}{l} + \frac{2\pi}{l}j\right).$$

Fixing an order l of coarse-graining, one can iterate this procedure n times. Indeed, taken a regular tiling of H by H' and recursively defining the n-th step as $T_{+1}^{(n)} := U_{H'}T_{+1}^{(n-1)}U_{H'}{}^\dagger$, one has

$$\begin{aligned}
T_{+1}^{(2)} &\equiv U_{H'}T_{+1}^{(1)}U_{H'}{}^\dagger \\
&= T_+^{H'} \otimes \left(|0\rangle\langle l-1|\right)^{\otimes 2} + T_e^{H'} \otimes \left(\sum_{j=0}^{l-1}|j+1\rangle\langle j| \otimes |0\rangle\langle l-1| + \mathbb{1} \otimes \sum_{j=0}^{l-1}|j+1\rangle\langle j|\right) \\
&= \int dk\,|k\rangle\langle k| \otimes \left(Z_l^+ \otimes |0\rangle\langle l-1| + \mathbb{1} \otimes \sum_{j=0}^{l-1}|j+1\rangle\langle j|\right) \\
&= \sum_j \int dk\,|k\rangle\langle k| \otimes \left[|j\rangle\langle j| \otimes \left(e^{+i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}|0\rangle\langle l-1| + \sum_{j'=0}^{l-1}|j'+1\rangle\langle j'|\right)\right]
\end{aligned}$$

$$= \sum_{j,j'} \int dk \, |k\rangle \langle k| \otimes |j\rangle \langle j| \otimes |j'\rangle \langle j'| \, e^{+i\left(\frac{k}{l^2}+\frac{2\pi}{l^2}j+\frac{2\pi}{l}j'\right)}, \tag{16}$$

and

$$T_{-1}^{(2)} \equiv U_{H'} T_{+1}^{(1)\dagger} U_{H'}{}^{\dagger} \equiv T_{-1}^{(2)\dagger}, \tag{17}$$

leading again to shifted and rescaled Weyl walk dispersion relation.
One can notice that the momenta are rescaled by a constant factor, depending on the order of regular tiling and of iteration; $k \in [-\pi, +\pi]$: one can therefore define a rescaled momentum through the change of variable $\tilde{k} = \frac{k}{l}$, with $\tilde{k} \in \left[\frac{-\pi}{l}, \frac{+\pi}{l}\right]$. This is a general feature of regularization, as it will seen in the next sections.

5.2 ONE-DIMENSIONAL DIRAC QW

We briefly recall the features of the Dirac QW, presented in chapter 3. The main differences with respect to the massless case are:

- The introduction of the identical transition in the position space, *i.e.* T_e associated with $A_e = im\sigma_x$ (the real parameter m is the mass of the walk).
- Due to the locality and unitarity of the coupling, the transition matrices are:

$$A_{+1} := \nu \, |+\rangle \langle +| , \quad A_{-1} := \nu \, |-\rangle \langle -| ,$$

with $m^2 + \nu^2 = 1$. $\tilde{T}_{\pm 1}$ are found as in the previous sections, while

$$\tilde{T}_e \equiv U_H T_e U_H^\dagger = T_e^H \otimes \mathbb{1}$$

trivially holds.
As before, one finds

$$\begin{aligned} \mathcal{R}[A] &= \int_{\mathscr{B}} dk \, |k\rangle \langle k| \otimes \left(Z_l^+ \otimes A_{+1} + Z_l^- \otimes A_{-1} + \mathbb{1} \otimes A_e\right) \\ &=: \int_{\mathscr{B}} dk \, |k\rangle \langle k| \otimes \mathcal{R}[A]_k \end{aligned}$$

and can simultaneously diagonalize the $Z_l^\pm$ and $\mathbb{1}$, finding

$$\mathcal{R}[A]_k = \sum_j e^{+i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)} |j\rangle \langle j| \otimes A_{+1} + \sum_j e^{-i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)} |j\rangle \langle j| \otimes A_{-1} + \sum_j \otimes A_e$$

$$= \sum_j |j\rangle\langle j| \otimes \underbrace{\left(e^{+i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}A_{+1} + e^{-i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)}A_{-1} + A_e\right)}_{:=\mathcal{D}(k,j)}.$$

The diagonal blocks in the previous equation represent just l Dirac walks with shifted and rescaled momenta:

$$\mathcal{D}(k,j) \equiv \begin{pmatrix} \nu e^{+i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)} & im \\ im & \nu e^{-i\left(\frac{k}{l}+\frac{2\pi}{l}j\right)} \end{pmatrix},$$

therefore the dispersion relation of the regularized walk is

$$\omega_{\pm}(k,j) = \pm \arccos\left\{\nu \cos\left[\frac{k}{l} + \frac{2\pi}{l}j\right]\right\}$$

Again, one can iterate this procedure, finding

$$T_e^{(2)} = T_e^{H'} \otimes \mathbb{1} \otimes \mathbb{1} = \sum_{j,j'} \int dk\, |k\rangle\langle k| \otimes |j\rangle\langle j| \otimes |j'\rangle\langle j'|\,;$$

then, using (16) and (17), this leads to an expression $\mathcal{D}(k,j,j')$ as before.

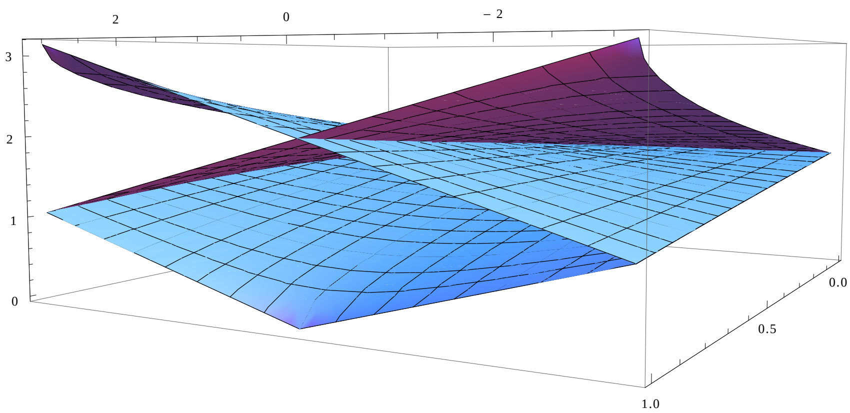

Figure 3: Plot of ω versus (k,ν), dispersion relation of the regularized 1D Dirac QW for $l = 3$ and one iteration step. One can notice the "folded" branches of the original dispersion relation (this was already exhibited in the 1D case, which was trivial from this point of view.). As m approaches to saturating the Dirac mass bound $m = 1$, ν approaches to zero and the dispersion relation tends to flatness.

5.3 TWO-DIMENSIONAL WEYL QW

For $d = 2$, the only massless unitary QW is found on $G \cong \mathbb{Z}^2$ and

$$G \sim \langle x, y \mid xyx^{-1}y^{-1} \rangle, \tag{18}$$

whose Cayley graph is just that whose vertices form the simple-cubic lattice.
Infinitely many types of tiling of G can be performed taking a subgroup $H \cong \mathbb{Z}^2$, but we will consider without loss of generality those achieved by $H = l\mathbb{Z} \times m\mathbb{Z}$. The other tilings are achieved simply by taking into account $H = H_1 \times H_2$ ($H_1, H_2 \cong \mathbb{Z}$), where the generators of H_1, H_2 are not proportional (in the sense of footnote on page 37) to those of G.
An advantage of grouping vertices into rectangular tiles is that this preserves the graph structure.
The 2D case is more interesting than the 1D one because of the existence of two independent directions, which can give rise to anisotropic coarse-grainings.

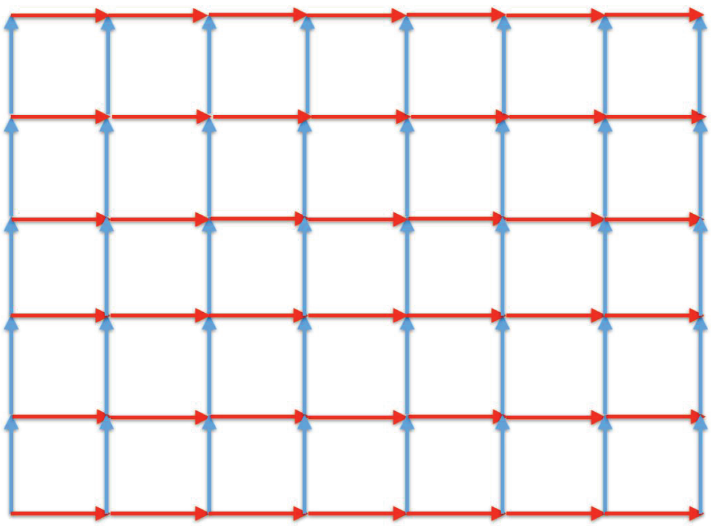

Figure 4: The Cayley graph of $\mathbb{Z}^2$ corresponding to the presentation (18).

The regular tilings of order $l \times m$ are defined as a partition

$$G = \bigcup_{i=0}^{l-1} \bigcup_{j=0}^{m-1} (ix + jx + H)\,;$$

G and H are isomorphic and identically presented, while the generators of H are expressed in terms of those of G as lx and my.
Clearly, due to the abelianity of the group, the tiling is invariant under the swapping of x and y (and accordingly of l and m).

By virtue of the regular tiling here defined, we can express the mapping operator as

$$U_H = \sum_{i,j} \sum_{x,y \in \mathbb{Z}} |x\rangle |i\rangle |y\rangle |j\rangle \langle lx + i| \langle my + j| .$$

U_H maps independently the generators of the two directions x, y, *i.e.*

$$\begin{aligned}\tilde{T}_{+x} &= \left(\sum_x |x+1\rangle \langle x| \otimes |0\rangle \langle l-1| + \sum_x |x\rangle \langle x| \otimes \sum_{i=1}^{l-1} |i+1\rangle \langle i| \right) \otimes \sum_{y,j} |y\rangle \langle y| \otimes |j\rangle \langle j| \\ &=: \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes X_l^+ \otimes \mathbb{1}_m ,\end{aligned}$$

$$\begin{aligned}\tilde{T}_{+y} &= \sum_{x,i} |x\rangle \langle x| \otimes |i\rangle \langle i| \left(\sum_y |y+1\rangle \langle y| \otimes |0\rangle \langle m-1| + \sum_y |y\rangle \langle y| \otimes \sum_{j=1}^{m-1} |j+1\rangle \langle j| \right) \\ &=: \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes \mathbb{1}_l \otimes Y_m^+ ,\end{aligned}$$

(and clearly $\tilde{T}_{-x} = \tilde{T}^\dagger_{+x}$, $\tilde{T}_{-y} = \tilde{T}^\dagger_{+y}$) where the x, y *shift matrices* have been defined analogously to the 1D case.
The 2D regularized walk thus reads:

$$\begin{aligned}\mathcal{R}[A] &= \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes X_l^+ \otimes \mathbb{1}_m \otimes A_{+x} + \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes X_l^- \otimes \mathbb{1}_m \otimes A_{-x} \\ &+ \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes \mathbb{1}_l \otimes Y_m^+ \otimes A_{+y} + \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes \mathbb{1}_l \otimes Y_m^- \otimes A_{-y} \\ &=: \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes \sum_{\mathbf{h} \in S} (B_h \otimes A_h) =: \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes \mathcal{R}[A]_{\mathbf{k}} .\end{aligned}$$

Simultaneously diagonalizing the B_h:

$$\begin{aligned}\mathcal{R}[A]_{\mathbf{k}} &= \sum_{i,j} \int_{\mathscr{B}} \mathbf{dk}\, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes |i\rangle \langle i| \otimes |j\rangle \langle j| \otimes \left(e^{+i\left(\frac{k_x}{l} + \frac{2\pi}{l} i\right)} A_{+x} + e^{-i\left(\frac{k_x}{l} + \frac{2\pi}{l} i\right)} A_{-x} \right. \\ &\left. + e^{+i\left(\frac{k_y}{m} + \frac{2\pi}{m} j\right)} A_{+y} + e^{-i\left(\frac{k_y}{m} + \frac{2\pi}{m} j\right)} A_{-y} \right) \equiv\end{aligned}$$

$$\equiv \sum_{i,j} \int_{\mathscr{B}} \mathrm{d}\mathbf{k} \, |\mathbf{k}\rangle \langle \mathbf{k}| \otimes |i\rangle \langle i| \otimes |j\rangle \langle j| \otimes \mathcal{W} \left(\frac{k_x}{l} + \frac{2\pi}{l} i, \frac{k_y}{m} + \frac{2\pi}{m} j \right), \tag{19}$$

namely $l \times m$ shifted and scaled Weyl QWs.
Iteration is straightforwardly performed in each direction separately as in the 1D case.

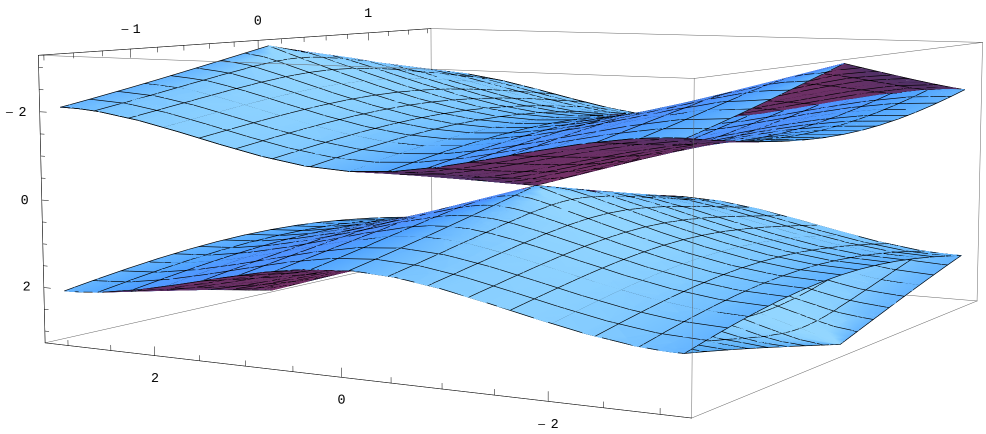

Figure 5: Plot of ω versus $(\tilde{k}_x, \tilde{k}_y)$, dispersion relation of the regularized 2D Weyl QW for the anisotropic regular tiling $l = 1, m = 2$ (one iteration step). One can notice the "folded" branches of the original dispersion relation around the y component only.

5.4 TWO-DIMENSIONAL DIRAC QW

The regularization of the 2D massive case is formally identical to the previous case examined.
Two Weyl walks are coupled and the transition matrices become

$$A_f \longrightarrow \begin{pmatrix} \nu A_f & 0 \\ 0 & \nu A_f^\dagger \end{pmatrix}, \quad A_e \longrightarrow \begin{pmatrix} 0 & im\mathbb{1} \\ im\mathbb{1} & 0 \end{pmatrix}.$$

The Dirac QW has the usual form

$$A = \sum_{f \in S} T_f \otimes A_f + T_e \otimes A_e,$$

so that one can regularize the massless part in the same way as in (19) by linearity, while the massive contribute reads

$$U_H (T_e \otimes A_e) U_H = \tilde{T}_e^H \otimes \mathbb{1}_l \otimes \mathbb{1}_m.$$

The Dirac regularized walk is obtained combining the 2D massless case with the derivation of the 1D massive walk, leading to the $l \times m$ shifted and rescaled dispersion relations

$$\omega_\pm (k_x, k_y, i, j) = \pm \arccos \left\{ \frac{\nu}{2} \left[\cos\left(\frac{k_x}{l} + \frac{2\pi}{l} i \right) + \cos\left(\frac{k_y}{m} + \frac{2\pi}{m} j \right) \right] \right\}.$$

5.5 DISCARDING THE ADDITIONAL DEGREE OF FREEDOM

In section 4.4 we gave a renormalization procedure valid for QWs over abelian groups, which consists in performing a regularization of the walk, applying the corresponding evolution operator to some state and then tracing out the additional degrees of freedom, in order to obtain a "coarse-grained" walk.
This procedure gives in general mixed states as a result: in [22] it is explained that "mixing can be always regarded as the result of discarding an environment, otherwise everything being describable in terms of pure states and reversible transformations" (this is entailed from the *postulate of purification* [19]). This fact is strictly connected with the *irreversibility* of the evolution. Accordingly, in general renormalizing a walk is equivalent to make it *irreversible*.

In view of this general feature, it is worth noticing that on one hand irreversible have been hardly studied so far; on the other hand, in the 1D massless case there exist states which preserve purity after having discarded the additional degree of freedom, namely the eigenstate of A in the momentum representation (which are the element of the canonical basis of $\mathbb{C}^2$).

Let's therefore take a state

$$\left|\psi^\pm\right\rangle = \sum_{x \in T} g(x) \left|x\right\rangle \left|\pm\right\rangle,$$

with T being a set collecting any site in the interior of some tile—provided it contains no site in the boundary of the tile under consideration—and $g(x)$ some

weight factors.
The regularized evolution of $|\psi\rangle$ is given by

$$\left|\psi'^{\pm}\right\rangle := \mathcal{R}\left[A\right] U_H \left|\psi^{\pm}\right\rangle = \mathcal{R}\left[A\right] |T\rangle \otimes \left(\sum_{j\in T} g(x_j)\,|j\rangle\right) \otimes |\pm\rangle\,,$$

and, since $A_\pm = |\pm\rangle\langle\pm|$, $\mathcal{R}\left[A\right]$ selects just one of the corresponding regularized translations $\tilde{T}_\pm$; moreover, as inferred from the form of (13) and (14), it performs a periodic shift on the elements of the additional space without selecting a shift on the *tile space*, thus the additional internal space remains factorized:

$$\left|\psi'^{\pm}\right\rangle \equiv |T\rangle\,|J\rangle\,|\pm\rangle\,.$$

$|J\rangle$ is normalized—since $|\psi\rangle$ is—and the factorization is a sufficient condition for the renormalization procedure to be reversible:

$$\mathrm{Tr}_2 \left|\psi'^{\pm}\right\rangle\left\langle\psi'^{\pm}\right| = |T\rangle\langle T| \otimes |\pm\rangle\langle\pm|\,.$$

6

RENORMALIZATION OF QWs OVER NON-ABELIAN GROUPS

6.1 QW OVER A VIRTUALLY ABELIAN GROUP

Let's consider the group G_1 of rank 2 endowed with the following presentation

$$G_1 \sim \langle a, b \mid a^2 b^{-2} \rangle. \tag{20}$$

The Cayley graph of G_1 is that whose vertices form the simple square lattice, quasi-isometrically embeddable in $\mathbb{R}^2$: indeed, as discussed in section 2.2, G_1 is virtually abelian.

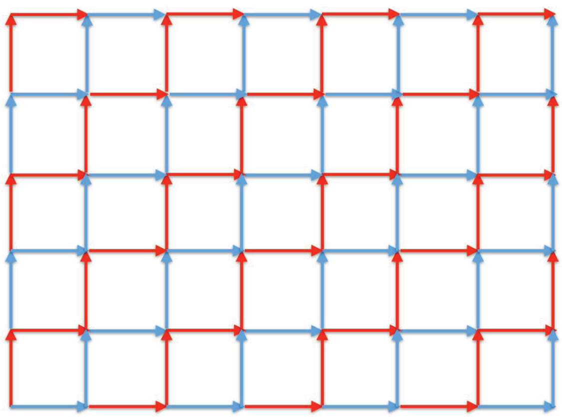

Figure 6: The Cayley graph of G_1 corresponding to the presentation (20).

One of its free abelian subgroups $H \cong \mathbb{Z}^2$ of finite index is generated by $h_2 = a^2$, $h_4 = a^{-1}b$. Some regular tessellations of the Cayley graph of G_1 are achieved by the cosets partitions

$$G_1 = H \cup fH, \quad f \in S_1 \tag{21}$$

namely H is of index 2. The sublattice induced by H is given by the following presentation

$$H \sim \langle h_1, h_2, h_3, h_4 \mid h_1 - (h_2 - h_4), \; h_3 - (h_2 + h_4) \rangle, \tag{22}$$

(abelian notation is used since $H \cong \mathbb{Z}^2$) which gives rise to a Cayley subgraph of degree 8 which tessellates the Cayley graph of G_1. In terms of the generators of G_1 one has $h_1 = ba$ and $h_3 = ab$.

In appendix A.1 it is derived the corresponding evolution operator B for the QW over the Cayley graph of G_1. The resulting transition matrices $\{A_q\}_{q \in S_1}$ are the same of the two-dimensional abelian case (see (8)), provided the following identifications:

$$a \leftrightarrow y, \; b \leftrightarrow x, \; a^{-1} \leftrightarrow -x, \; b^{-1} \leftrightarrow -y \tag{23}$$

Due to the normality of H in G_1, in view of the remark 4.3 we can define the invariant spaces under the T_h:

$$|\mathbf{k}\rangle_0 := \frac{1}{2\pi} \sum_{\mathbf{x} \in H} e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}\rangle, \quad |\mathbf{k}\rangle_1 := \frac{1}{2\pi} \sum_{\mathbf{x} \in H} e^{-i\mathbf{k}\cdot\mathbf{x}} |f\mathbf{x}\rangle.$$

Evaluating the action of the generator of G_1 on the $|\mathbf{k}\rangle_j$, one can reconstruct the renormalized non abelian walk $\mathcal{R}[B]$, which will be written in terms of the 8 generators of H and their inverses. In order to do so, we have now to fix a regular tiling of the Cayley graph of G_1, *e.g.* the one with $f = a$; from (21), however, it is interesting to note that the action of the generators of G_1 and their inverses on a left coset of H takes values in the other, because $f^2 \in H \; \forall f$ and of course $fH \equiv aH$ for disjointness of cosets. This fact is useful in order to write an expression for $\mathcal{R}[B]$.
One has:

$$T_a |\mathbf{k}\rangle_0 = |\mathbf{k}\rangle_1, \qquad T_a |\mathbf{k}\rangle_1 = e^{ik_2} |\mathbf{k}\rangle_0,$$
$$T_b |\mathbf{k}\rangle_0 = e^{ik_4} |\mathbf{k}\rangle_1, \qquad T_b |\mathbf{k}\rangle_1 = e^{ik_1} |\mathbf{k}\rangle_0,$$

$$T_{a^{-1}}|\mathbf{k}\rangle_0 = e^{-ik_2}|\mathbf{k}\rangle_1\,, \qquad T_{a^{-1}}|\mathbf{k}\rangle_1 = |\mathbf{k}\rangle_0\,,$$
$$T_{b^{-1}}|\mathbf{k}\rangle_0 = e^{-ik_1}|\mathbf{k}\rangle_1\,, \qquad T_{b^{-1}}|\mathbf{k}\rangle_1 = e^{-ik_4}|\mathbf{k}\rangle_0\,.$$

It follows this off-diagonal expression for the renormalized walk

$$\mathcal{R}[B]_{\mathbf{k}} = \begin{pmatrix} 0 & B_{\mathbf{k}} \\ B'_{\mathbf{k}} & 0 \end{pmatrix},$$

with

$$\begin{aligned} B_{\mathbf{k}} &= e^{ik_2}A_a + e^{ik_1}A_b + A_{a^{-1}} + e^{-ik_4}A_{b^{-1}}, \\ B'_{\mathbf{k}} &= A_a + e^{ik_4}A_b + e^{-ik_2}A_{a^{-1}} + e^{-ik_2}A_{b^{-1}}. \end{aligned} \tag{24}$$

Exploiting the relators in (22) and the isotropy group representation derived in A.1, equations (24) read

$$\begin{aligned} B_{\mathbf{k}} &= e^{i\frac{k_1}{2}}\left(e^{i\frac{k_3}{2}}A_a + e^{i\frac{k_1}{2}}A_b + e^{-i\frac{k_1}{2}}A_{a^{-1}} + e^{-i\frac{k_3}{2}}A_{b^{-1}}\right) := e^{i\frac{k_1}{2}}\mathcal{W}_1, \\ B'_{\mathbf{k}} &= e^{-i\frac{k_1}{2}}\left(e^{i\frac{k_1}{2}}A_a + e^{i\frac{k_3}{2}}A_b + e^{-i\frac{k_3}{2}}A_{a^{-1}} + e^{-i\frac{k_1}{2}}A_{b^{-1}}\right) := e^{-i\frac{k_1}{2}}\sigma_z\mathcal{W}_1\sigma_z, \end{aligned}$$

Defining

$$V := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & e^{-i\frac{k_1}{2}}\sigma_z \end{pmatrix},$$

one has

$$\mathcal{R}[B]_{\mathbf{k}} = V\begin{pmatrix} 0 & \mathcal{W}_1\sigma_z \\ \mathcal{W}_1\sigma_z & 0 \end{pmatrix}V^\dagger \equiv V\left(\sigma_x \otimes \mathcal{W}_1\sigma_z\right)V^\dagger.$$

Accordingly, the eigenvalues of $\mathcal{R}[B]_{\mathbf{k}}$ are those of $\sigma_x \otimes \mathcal{W}_1\sigma_z$; since $\sigma_x = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\mathcal{R}[B]_{\mathbf{k}}$ is unitarily equivalent to the block diagonal operator

$$\begin{pmatrix} \mathcal{W}_1\sigma_z & 0 \\ 0 & -\mathcal{W}_1\sigma_z \end{pmatrix}.$$

Resorting to A.1 and to the correspondence (23), it's easy to see that

$$i\mathcal{W}_1\sigma_z \equiv \mathcal{W}\left(\frac{k_1+\pi}{2}, \frac{k_3+\pi}{2}\right),$$

thus the four eigenvalues of $\mathcal{R}\,[B]_{\mathbf{k}}$ are

$$\pm\lambda^{+}, \quad \pm\lambda^{-} \quad \text{with} \quad \lambda^{\pm} := e^{\pm i \arccos\left\{\frac{1}{2}\left[\cos\frac{k_1+\pi}{2} + \cos\frac{k_3+\pi}{2}\right]\right\} - i\frac{\pi}{2}},$$

namely the same of the Weyl QW up to some shifts for the momenta and the energy.

Up to now we worked on the set I of transition matrices found in A.1, leading to $\mathcal{R}\,[B]_{\mathbf{k}} \equiv \mathcal{R}\left[B^{I}\right]_{\mathbf{k}}$; in A.1 two sets of matrices are actually found and it is interesting to investigate if they are connected in some physically meaningful way.
Through a block diagonal change of basis matrix, we see that $\mathcal{R}\left[B^{I}\right]_{\mathbf{k}}$ is unitarily equivalent to

$$\begin{pmatrix} 0 & \sigma_z \mathcal{W}_1 \\ \sigma_z \mathcal{W}_1 & 0 \end{pmatrix}.$$

On the other hand, one finds in A.1 that B^{II} is connected to B^{I} through a local antiunitary transformation on the A_q^{II}, therefore $\mathcal{R}\left[B^{II}\right]_{\mathbf{k}}$ is, by linearity, unitarily equivalent to

$$\begin{pmatrix} 0 & \mathcal{W}_1^t \sigma_z \\ \mathcal{W}_1^t \sigma_z & 0 \end{pmatrix},$$

which is just $\mathcal{R}\left[B^{I}\right]_{-\mathbf{k}}^{\dagger}$ (up to a change of basis).
This means that the two *regularized walks* are connected by PT symmetry, where

$$\begin{aligned} \mathsf{P} &: \mathbf{k} \longmapsto -\mathbf{k} \quad \text{(parity)}, \\ \mathsf{T} &: A \longmapsto A^{\dagger} \quad \text{(time reversal)}. \end{aligned}$$

This physical interpretation of P symmetry can be attempted in view of the fact that the non-abelian walk has been regularized, recovering an abelian walk and enabling to define significantly physical quantities (such as the momentum).

6.2 QW OVER A GROUP WITH CYCLIC GENERATORS

Let's consider the group G_2 of rank 2 endowed with the following presentation

$$G_2 \sim \langle a, b \mid a^4, b^4, (ab)^2 \rangle. \tag{25}$$

Also in this case the Cayley graph of G_2 is that whose vertices form the simple square lattice and G_2 is virtually abelian; one of its free abelian subgroups of finite index $H \cong \mathbb{Z}^2$ is generated by $h_1 = b^{-1}a$, $h_2 = ab^{-1}$.

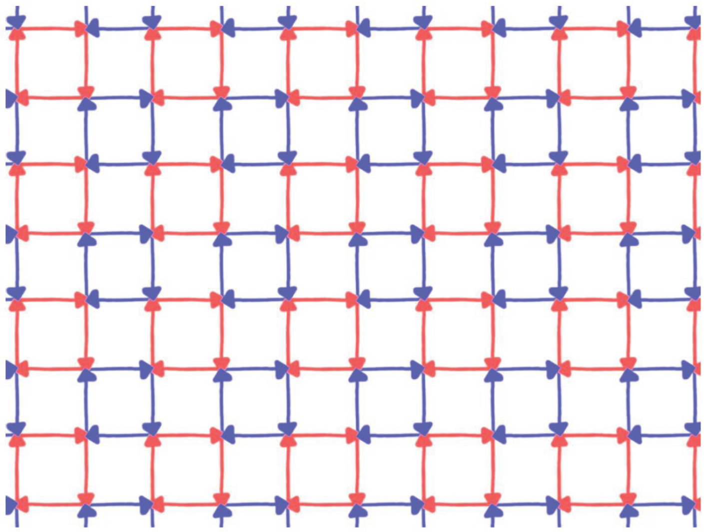

Figure 7: The Cayley graph of G_2 corresponding to the presentation (25).

Some regular tessellations of the Cayley graph of G_2 are achieved by the cosets partition

$$G_2 = \bigcup_{j=0}^{3} f^j H, \quad f \in S_2. \tag{26}$$

The sublattice induced by H in G_2 gives rise to a simple square Cayley subgraph that tessellates the Cayley graph of G_2.
This time the subgroup H is not normal in G_2, hence it is not possible to define the invariant spaces under the T_h.

In appendix A.2 it is derived the corresponding evolution operator C for the QW over the Cayley graph of G_2. Unlike the case of G_1, the resulting transition matrices $\{A_q\}_{q \in S_2}$ are different from the two-dimensional abelian case.
Let's fix a regular tiling, *e.g.* $f = a$, and define the spaces:

$$|\mathbf{k}\rangle_j := \frac{1}{2\pi} \sum_{\mathbf{x} \in H} e^{-i\mathbf{k}\cdot\mathbf{x}} \left|a^j \mathbf{x}\right\rangle, \quad j = 0, 1, 2, 3.$$

As in the previous case, in order to find $\mathcal{R}[C]$ one has to compute:

$$
\begin{aligned}
T_a\,|\mathbf{k}\rangle_0 &= |\mathbf{k}\rangle_1\,, & T_{a^{-1}}\,|\mathbf{k}\rangle_0 &= |\mathbf{k}\rangle_3\,,\\
T_a\,|\mathbf{k}\rangle_1 &= |\mathbf{k}\rangle_2\,, & T_{a^{-1}}\,|\mathbf{k}\rangle_1 &= |\mathbf{k}\rangle_0\,,\\
T_a\,|\mathbf{k}\rangle_2 &= |\mathbf{k}\rangle_3\,, & T_{a^{-1}}\,|\mathbf{k}\rangle_2 &= |\mathbf{k}\rangle_1\,,\\
T_a\,|\mathbf{k}\rangle_3 &= |\mathbf{k}\rangle_0\,, & T_{a^{-1}}\,|\mathbf{k}\rangle_3 &= |\mathbf{k}\rangle_2\,,\\
T_b\,|\mathbf{k}\rangle_0 &= e^{-ik_1}\,|\mathbf{k}\rangle_1\,, & T_{b^{-1}}\,|\mathbf{k}\rangle_0 &= e^{ik_2}\,|\mathbf{k}\rangle_3\,,\\
T_b\,|\mathbf{k}\rangle_1 &= e^{ik_2}\,|\mathbf{k}\rangle_2\,, & T_{b^{-1}}\,|\mathbf{k}\rangle_1 &= e^{ik_1}\,|\mathbf{k}\rangle_0\,,\\
T_b\,|\mathbf{k}\rangle_2 &= e^{ik_1}\,|\mathbf{k}\rangle_3\,, & T_{b^{-1}}\,|\mathbf{k}\rangle_2 &= e^{-ik_2}\,|\mathbf{k}\rangle_1\,,\\
T_b\,|\mathbf{k}\rangle_3 &= e^{-ik_2}\,|\mathbf{k}\rangle_0\,, & T_{b^{-1}}\,|\mathbf{k}\rangle_3 &= e^{-ik_1}\,|\mathbf{k}\rangle_2\,.
\end{aligned}
$$

It follows this block expression for the renormalized walk

$$
\mathcal{R}[C]_{\mathbf{k}} = \begin{pmatrix} C_{k_1} & C'_{k_2} \\ C'_{-k_2} & C_{-k_1} \end{pmatrix},
$$

with

$$
\begin{aligned}
C_k &= \begin{pmatrix} 0 & A_{a^{-1}} + e^{ik}A_{b^{-1}} \\ A_a + e^{-ik}A_b & 0 \end{pmatrix},\\
C'_k &= \begin{pmatrix} 0 & A_a + e^{-ik}A_b \\ A_{a^{-1}} + e^{-ik}A_{b^{-1}} & 0 \end{pmatrix}.
\end{aligned}
\tag{27}
$$

In order to find the eigenvalues of $\mathcal{R}[C]_{\mathbf{k}}$ for the case I of appendix A.2, we compute $\left(\mathcal{R}\left[C^{I}\right]_{\mathbf{k}}\right)^2$, namely:

$$
\begin{pmatrix} C_{k_1}^2 + C'_{k_2}C'_{-k_2} & C_{k_1}C'_{k_2} + C'_{k_2}C_{-k_1} \\ C'_{-k_2}C_{k_1} + C_{-k_1}C'_{-k_2} & C'_{-k_2}C'_{k_2} + C^2_{-k_1} \end{pmatrix}.
$$

Since $C_kC'_{k'} = -C'_{k'}C_{-k}$, the off-diagonal blocks vanish; moreover, both the diagonal blocks are equal to

$$
2\zeta^{\pm}\zeta^{\mp}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.
$$

Accordingly, the eigenvalues of $\mathcal{R}\left[C^{I}\right]_{\mathbf{k}}$ are just $\left(2\zeta^{\pm}\zeta^{\mp}\right)^{\frac{1}{2}} \equiv 1, -1$.
The peculiar feature of this QW is that $\omega_j \neq \omega_j(\mathbf{k})$, so $\mathcal{R}\left[C^{I}\right]_{\mathbf{k}}$ is an abelian walk which lead to a non dispersive Schödinger equation (see the footnote on page 23). The factors $\zeta^{\pm}$ are not relevant in order to distinguish the walks in some relevant way.

The case II is not unitarily equivalent to the I:

$$\mathcal{R}\left[C^{II}\right]_{\mathbf{k}} = \begin{pmatrix} D_{k_1} & D'_{k_2} \\ D'_{-k_2} & D_{-k_1} \end{pmatrix},$$

with

$$D_k = \begin{pmatrix} 0 & \sigma_x\left(A^{I}_{a^{-1}} + e^{ik}A^{I}_{b^{-1}}\right)\sigma_x \\ A^{I}_{a} + e^{-ik}A^{I}_{b} & 0 \end{pmatrix},$$
$$D'_k = \begin{pmatrix} 0 & A^{I}_{a} + e^{-ik}A^{I}_{b} \\ \sigma_x\left(A^{I}_{a^{-1}} + e^{-ik}A^{I}_{b^{-1}}\right)\sigma_x & 0 \end{pmatrix}.$$

Numerical analysis of the eigenvalues of $\mathcal{R}\left[C^{II}\right]_{\mathbf{k}}$ gives the usual large scale limit to the two-dimensional Weyl dispersion relation $\omega(\mathbf{k}) = \sqrt{k_1^2 + k_2^2}$.
Resorting to remark A.4, instead, case I can "transit" through different interesting behaviours, as shown and discussed in the figures below.

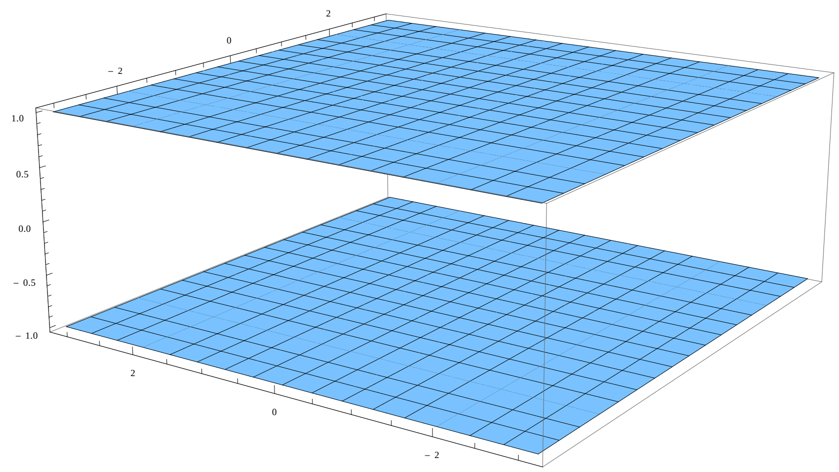

Figure 8: Case already discussed in the present section: $b = 0$, $U_{\pm} = \mathbb{1}$, which gives a flat dispersion relation.

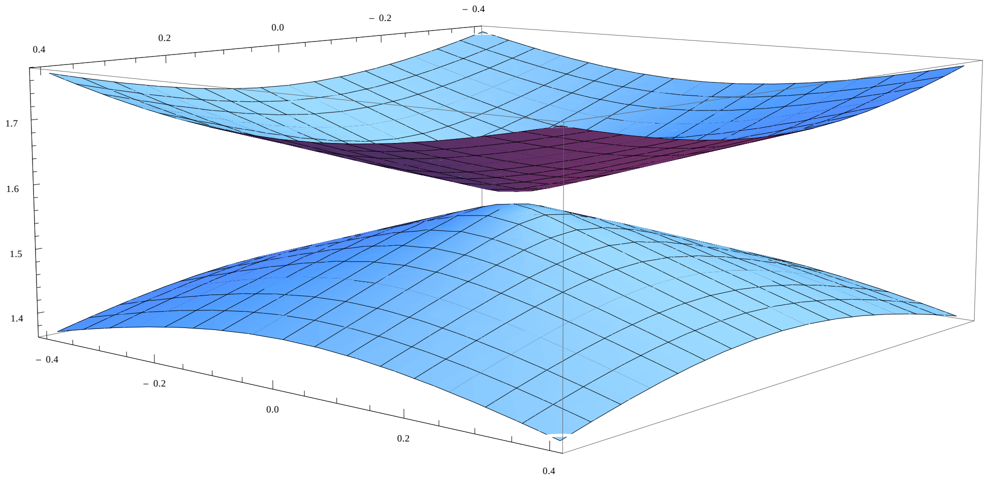

Figure 9: Intermediate case: mass effect. The two cones of the dispersion relation are detached. "Mass parameter" $a = 0.01$, $b = \sqrt{1-a^2}$. The Dirac dispersion relation tends to flatness for ν approaching to zero, *i.e.* m approaching to 1 (here $\nu \to b$, $m \to a$), as seen in figure 3 on page 41.

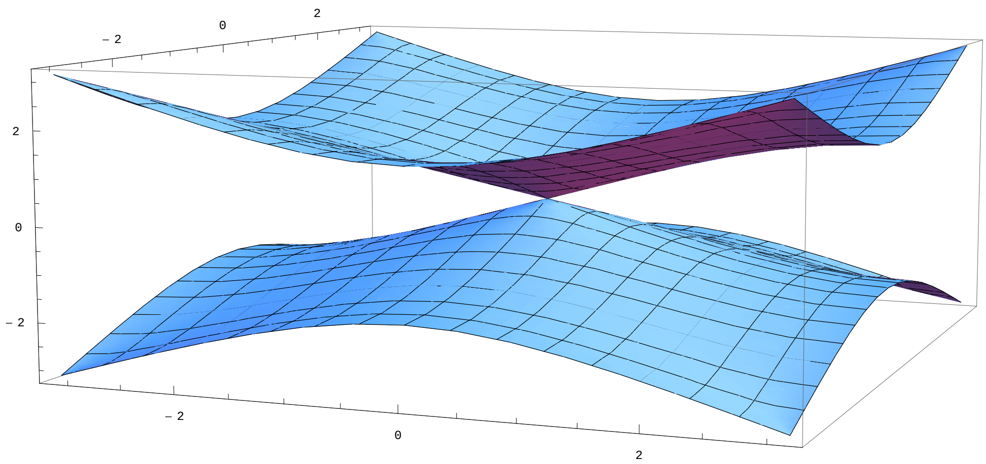

Figure 10: Weyl transition: $a = 0$ and $U_- = -i\sigma_x$, as expected in view of the analogy noticed above.

CONCLUSIONS

La contingenza, che fuor del quaderno
de la vostra matera non si stende,
tutta è dipinta nel cospetto etterno;
necessità però quindi non prende
se non come dal viso in che si specchia
nave che per torrente giù discende.

Dante Alighieri [Divina Commedia, Paradiso, Canto XVII]

In this work we have studied the dynamical model known as random walk.
The applicative scopes of RWs are extensive and have been largely explored during the 19th century.
RWs have been extended to the quantum case: a quantum walk (QW) shows even more interesting emergent behaviours due to the interference of the quantum paths.
In the last two decades they have been also taken into account as the fundamental mechanism governing systems at ultra-relativistic energies, namely at Planck scale, where violations of Lorentz symmetry are hypothesised.

We have studied a peculiar model in this purview, namely QWs over Cayley graphs, with *homogeneity* and *locality* of interactions as main requirements.
This model have been studied by the autors of [40, 52, 53] in the perspective of re-founding *Quantum Field Theory* and it gives emergent Weyl and Dirac evolution in the large-scale limit.
The thought-provoking idea of the discreteness of spacetime at a fundamental

level is grounded on informational and operational principles.
In [54] it is shown how this kind of walk exhibits *relative locality*, a feature of *doubly-special relativity* (which is a deformed Lorentz symmetry theorized by Amelino-Camelia in [55]).

In this thesis work we firstly considered abelian QWS. We found a closed-form expression for renormalizing the walks in the position space; we studied the resulting dispersion relations, which have been found to be invariant with respect to the non-regularized case (up to some shifts and rescalings).
Subsequently, we took the large-scale limit performing a coarse-graining of the quantum network, finding that in general this gives rise to an *irreversible* walk, as expected. A *reversible* (*i.e.* preserving unitarity) evolution can be found in the Weyl 1D case.
Through a group-theoretical treatment, we have been able to provide a rigorous renormalization of non-abelian QWS, furnishing the first examples of this kind of walks. In the non-abelian case the regularization procedure becomes crucial, as discussed during the thesis.
The dispersion relations of the non-abelian walks studied exhibit features that are similar to the abelian case. Besides, three peculiar issues have to be pointed out:

- Although we studied a group with cyclic conditions resorting to an abelian version of it, we founded a flat dispersion relation, meaning that the properties of the original group are inherited by the procedure of renormalization. This fact attests the correctness of the framework and entails that abelian walks can exhibit a variety of behaviour (provided the extension of the *cell structure*).
- The concept of *strong homogeneity* (mentioned in 3.1) is an emergent feature in this framework.
- Mass effects can be obtained with this procedure without coupling two Weyl walks.

6.3 FUTURE PERSPECTIVES

There is still room for exploring new features of both the model and of the renormalization framework.

Non-abelian QW may be a fruitful subject of study. The class satisfying the physical assumptions is fixed (virtually abelian groups) and proved to give quite different behaviours with respect to the abelian class. Further investigations are possible also for the 3D case, which has been explored for $G \cong \mathbb{Z}^3$ only.
In literature, *irreversible* QWs (appeared renormalizing the abelian walks) have been hardly studied. Moreover, in [39] "the operational procedure of building up the coordinate system introduces an in-principle indistinguishability between neighboring events, resulting in a network that is coarse-grained": this is analogous to the coarse-graining of the walk's lattice, making potentially interesting to study *boosts* also in this context.
Lastly, a thorough characterization of the admissible graphs (and of the related QWs over them) might hopefully allow to comprehend whether Dirac dynamics (apart from non-dispersive cases) is an emergent general feature from a geometric point of view.

Part III

APPENDIX

A

DERIVATION OF QWs OVER NON-ABELIAN GROUPS

A.1 FIRST EXAMPLE OF A VIRTUALLY ABELIAN QW

Let's take the group G_1 endowed with the following presentation

$$G_1 \sim \langle a, b \mid a^2 b^{-2} \rangle.$$

We solve the unitarity conditions for $B = \sum_{q \in S_1} T_f \otimes A_q$ acting on $\ell^2(G_1) \otimes \mathbb{C}^2$.

Taking i, j assuming values in S_1^+ with $i \neq j$, from the conditions (3), one obtains the sets of equations:

$$A_i A_{j^{-1}}{}^\dagger = A_i{}^\dagger A_{j^{-1}} = 0 \tag{28}$$

and

$$\begin{aligned} A_i A_j{}^\dagger + A_{i^{-1}} A_{j^{-1}}{}^\dagger = A_i{}^\dagger A_j + A_{i^{-1}}{}^\dagger A_{j^{-1}} = 0 \\ A_i A_{i^{-1}}{}^\dagger + A_j A_{j^{-1}}{}^\dagger = A_i{}^\dagger A_{i^{-1}} + A_j{}^\dagger A_{j^{-1}} = 0 \end{aligned} \tag{29}$$

Every complex square matrix admits the so called *polar decomposition*, namely

$$\begin{aligned} \forall M \in \mathcal{M}(2 \times 2, \mathbb{C}) \ \exists\, V \in \mathcal{M}(2 \times 2, \mathbb{C}) \ \text{unitary}, \\ P \in \mathcal{M}(2 \times 2, \mathbb{C}) \ \text{semi-positive definite} \ : \ M = VP \end{aligned}$$

Taking a polar decomposition $A_q = V_q P_q$ ($q \in S_1$) and considering that $P_q = P_q^\dagger$, equations (28) become

$$V_i P_i P_{j^{-1}} {V_{j^{-1}}}^\dagger = 0 \tag{30}$$

$$P_i {V_i}^\dagger V_{j^{-1}} P_{j^{-1}} = 0 \tag{31}$$

Resorting to the Sylvester rank inequality for 2×2 matrices, which states $\mathrm{rk}D + \mathrm{rk}E \leqslant \mathrm{rk}DE + 2$, equations (30) imply that $\mathrm{rk}P_i + \mathrm{rk}P_{j^{-1}} \leqslant 2$: none P_q can be full rank unless some $P_{q'}$ is null rank, but this is excluded by the isotropy assumption. Then $\mathrm{rk}P_q = 1$. Furthermore, still from (30), one can write:

$$P_i = \alpha_i \left|+_i\right\rangle \left\langle+_i\right|, \quad P_{j^{-1}} = \alpha_{j^{-1}} \left|-_i\right\rangle \left\langle-_i\right|$$

with $\alpha_q > 0$ and $\{\left|+_i\right\rangle, \left|-_i\right\rangle\}$ orthonormal bases for $\mathbb{C}^2$.

On the other hand, from equations (31) one finds that $\left\langle+_i\right| {V_i}^\dagger V_{j^{-1}} \left|-_i\right\rangle = 0$ and, since ${V_i}^\dagger V_{j^{-1}}$ is unitary, it must be diagonal on the basis $\{\left|+_i\right\rangle, \left|-_i\right\rangle\}$ (and its entries are phase factors for unitarity).
Moreover, the V_q are not uniquely determined by the polar decomposition, since the A_q are not full rank.
In fact, if $A_i = V_i P_i$ holds for a V_i, there exists an infinite class of unitary matrices V_i' such that $A_i = V_i' P_i$. Let's take *e.g.* the matrix of the form

$$V_i' = V_i \left(\left|+_i\right\rangle \left\langle+_i\right| + e^{i\theta_i} \left|-_i\right\rangle \left\langle-_i\right| \right);$$

this is unitary and exactly observes the polar decomposition for A_i. In view of the fact that the same considerations hold for $A_{j^{-1}}$, one can always fix this "gauge freedom" computing

$${V_i'}^\dagger V_{j^{-1}}' = e^{i\theta_{j^{-1}}} \left\langle+_i\right| {V_i}^\dagger V_{j^{-1}} \left|+_i\right\rangle \left|+_i\right\rangle \left\langle+_i\right| + e^{-i\theta_i} \left\langle-_i\right| {V_i}^\dagger V_{j^{-1}} \left|-_i\right\rangle \left|-_i\right\rangle \left\langle-_i\right|$$

and arbitrarily posing $e^{i\theta_{j^{-1}}} \left\langle+_i\right| {V_i}^\dagger V_{j^{-1}} \left|+_i\right\rangle = 1$ and $e^{-i\theta_i} \left\langle-_i\right| {V_i}^\dagger V_{j^{-1}} \left|-_i\right\rangle = 1$, so that

$${V_i'}^\dagger V_{j^{-1}}' = \mathbb{1} \implies V_{j^{-1}}' = V_i'$$

holds.
This allows to write:

$$\begin{aligned} A_a &= \alpha_a V \left|+_a\right\rangle \left\langle+_a\right|, \quad A_{b^{-1}} = \alpha_{b^{-1}} V \left|-_a\right\rangle \left\langle-_a\right|, \\ A_b &= \alpha_b W \left|+_b\right\rangle \left\langle+_b\right|, \quad A_{a^{-1}} = \alpha_{a^{-1}} W \left|-_b\right\rangle \left\langle-_b\right|. \end{aligned} \tag{32}$$

Equations (29), exploiting the (28), then imply

$$\begin{aligned} A_i {A_j}^\dagger A_i &= 0 \\ A_i {A_{i^{-1}}}^\dagger A_i &= 0 \\ A_{i^{-1}} {A_{j^{-1}}}^\dagger A_{i^{-1}} &= 0 \\ A_{i^{-1}} {A_i}^\dagger A_{i^{-1}} &= 0 \end{aligned}$$

Inserting equations (32), these are equivalent to

$$\begin{aligned} \left\langle+_a\middle| -_b\right\rangle \left\langle-_b\right| W^\dagger V \left|+_a\right\rangle &= 0, \\ \left\langle-_a\middle| +_b\right\rangle \left\langle+_b\right| W^\dagger V \left|-_a\right\rangle &= 0, \\ \left\langle+_a\middle| +_b\right\rangle \left\langle+_b\right| W^\dagger V \left|+_a\right\rangle &= 0, \\ \left\langle-_a\middle| -_b\right\rangle \left\langle-_b\right| W^\dagger V \left|-_a\right\rangle &= 0. \end{aligned} \tag{33}$$

Considering that $\left\langle+_a\middle| -_b\right\rangle = 0 \Leftrightarrow \left|+_a\right\rangle = \left|+_b\right\rangle \Leftrightarrow \left\langle-_a\middle| +_b\right\rangle = 0$ (up to phase factors that would not appear in the A_q), for the (33) to be satisfied there are two cases only:

I

$$\left.\begin{aligned} \left|+_a\right\rangle &= \left|+_b\right\rangle \\ \left|-_a\right\rangle &= \left|-_b\right\rangle \end{aligned}\right\} \implies \left\langle+_b\right| W^\dagger V \left|+_a\right\rangle = \left\langle-_b\right| W^\dagger V \left|-_a\right\rangle = 0$$

II

$$\left\langle-_b\right| W^\dagger V \left|+_a\right\rangle = \left\langle+_b\right| W^\dagger V \left|-_a\right\rangle = 0$$

Let's note that anyhow just two of the matrix elements which appear in (33) can be zero: indeed, if *ab absurdo* this would not be the case, let's define U any of the possible matrices which connect the two orthonormal bases found; thus $U^\dagger W^\dagger V$ would have at least three vanishing matrix elements, but this is absurd for it is unitary.
Accordingly, the two cases are

I

$$\left.\begin{aligned} |+_a\rangle = |+_b\rangle := |+\rangle \\ |-_a\rangle = |-_b\rangle := |-\rangle \end{aligned}\right\} \implies \langle +| W^\dagger V |+\rangle = \langle -| W^\dagger V |-\rangle = 0$$

II

$$\left.\begin{aligned} |+_a\rangle = |-_b\rangle := |+\rangle \\ |-_a\rangle = |+_b\rangle := |-\rangle \end{aligned}\right\} \implies \langle +| W^\dagger V |+\rangle = \langle -| W^\dagger V |-\rangle = 0$$

In both cases one thus has $V := W(\mu |+\rangle\langle -| + \nu |-\rangle\langle +|)$.
Plugging these results in (29), in both cases one finds:

$$\left.\begin{aligned} \alpha_a \alpha_{a^{-1}} &= -\alpha_b \alpha_{b^{-1}} \nu^* \mu^* \\ \alpha_a \alpha_b &= -\alpha_{a^{-1}} \alpha_{b^{-1}} \nu \mu \end{aligned}\right\} \implies \mu\nu = -1,\ \alpha_{a^{-1}} = \alpha_b := \beta,\ \alpha_a = \alpha_{b^{-1}} := \alpha$$

namely

I

$$\begin{aligned} A_a &= \alpha\nu W |-\rangle\langle +| \\ A_b &= \beta W |+\rangle\langle +| \\ A_{a^{-1}} &= \beta W |-\rangle\langle -| \\ A_{b^{-1}} &= -\alpha\nu^* W |+\rangle\langle -| \end{aligned}$$

II

$$\begin{aligned} A_a &= \alpha\nu W |-\rangle\langle +| \\ A_b &= \beta W |-\rangle\langle -| \\ A_{a^{-1}} &= \beta W |+\rangle\langle +| \\ A_{b^{-1}} &= -\alpha\nu^* W |+\rangle\langle -| \end{aligned}$$

and the two cases are connected through the simple swap $b \leftrightarrow a^{-1}$.
From the normalization (4) it straightforwardly follows that $\beta = \sqrt{1-\alpha^2}$, while from the isotropy condition (5), W is found simply substituting the A_q and inverting the resulting relation, leading to

$$W_{\mathrm{I}} = \begin{pmatrix} \sqrt{1-\alpha^2} & \alpha\nu^* \\ -\alpha\nu & \sqrt{1-\alpha^2} \end{pmatrix}.$$

Finally, the transition matrices for case I are:

$$A_a^{I} = \alpha \begin{pmatrix} \alpha & 0 \\ \sqrt{1-\alpha^2}\nu & 0 \end{pmatrix}, \qquad A_b^{I} = \sqrt{1-\alpha^2} \begin{pmatrix} \alpha & 0 \\ -\alpha\nu & 0 \end{pmatrix},$$
$$A_{a^{-1}}^{I} = \sqrt{1-\alpha^2} \begin{pmatrix} 0 & \alpha\nu^* \\ 0 & \alpha \end{pmatrix}, \qquad A_{b^{-1}}^{I} = \alpha \begin{pmatrix} 0 & -\sqrt{1-\alpha^2}\nu^* \\ 0 & \alpha \end{pmatrix}.$$

As one can easily verify, a unitary matrix X such that $XA_a^{I}X^\dagger = A_b^{I}$ must be diagonal and so it would multiply the entries of A_a^{I} by a phase factor; accordingly, the transitive action 2 of the isotropy group on S_1^+ imposes $\alpha^2 = 1 - \alpha^2$, namely $\alpha = \frac{1}{\sqrt{2}}$.

Let's define the unitary $U(\nu) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu \end{smallmatrix}\right)$: by direct computation, one can verify that $U(\nu) A_q(\nu) U(\nu)^\dagger = A_q(1)$. By linearity of B, this implies that with a local unitary conjugation one can remove the dependence from the phase factor ν: the transition matrices I are then the same (up to a permutation of indexes) found in [40], namely:

$$A_a^{I} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_b^{I} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$$
$$A_{a^{-1}}^{I} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_{b^{-1}}^{I} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Remark A.1. *Case I and case II are connected by an antiunitary transformation; explicitly, it holds*

$$Y\left(A_q^{I}\right)^t Y^\dagger = A_q^{II}, \quad Y = \frac{1}{\sqrt{2}}\left(\mathbb{1} + i\sigma_y\right).$$

In section 6.1 it is provided the physical interpretation of this connection.
The unitary matrix which represents the action of the isotropy group of B is σ_z.

Remark A.2. *While, though it does not exist a unitary transformation between the two sets of matrices found for B, but they are the same up to an antiunitary transformation, yet it exists neither a unitary nor an antiunitary which connects the abelian set with the non abelian one.*

A.2 SECOND EXAMPLE OF A VIRTUALLY ABELIAN QW

Let's now consider a group of rank 2 with cyclic generators, namely

$$G_2 \sim \langle a, b \mid a^4, b^4, (ab)^2 \rangle.$$

We solve the unitarity conditions for $C = \sum_{q \in S_2} T_q \otimes A_q$ acting on $\ell^2(G_2) \otimes \mathbb{C}^2$.

Taking i, j assuming values in S_2^+ with $i \neq j$, from the conditions (3), one obtains the sets of equations:

$$\begin{aligned} A_{i^{-1}} {A_{j^{-1}}}^\dagger = {A_{i^{-1}}}^\dagger A_{j^{-1}} = 0 \\ A_i {A_j}^\dagger = {A_i}^\dagger A_j = 0 \end{aligned} \tag{34}$$

and

$$\begin{aligned} A_i {A_{i^{-1}}}^\dagger + A_{i^{-1}} {A_i}^\dagger = {A_{i^{-1}}}^\dagger A_i + {A_i}^\dagger A_{i^{-1}} = 0 \\ A_i {A_{j^{-1}}}^\dagger + A_{j^{-1}} {A_i}^\dagger = {A_{i^{-1}}}^\dagger A_j + {A_j}^\dagger A_{i^{-1}} = 0 \end{aligned} \tag{35}$$

Taking a polar decomposition for the A_q similarly to what has been done in A.1, the following form of the transition matrices follows:

$$A_a = \alpha_a V \left|+_a\right\rangle \left\langle +_a\right|, \qquad A_b = \alpha_b V \left|-_a\right\rangle \left\langle -_a\right|, \tag{36}$$

$$A_{a^{-1}} = \alpha_{a^{-1}} W \left|+_{a^{-1}}\right\rangle \left\langle +_{a^{-1}}\right|, \qquad A_{b^{-1}} = \alpha_{b^{-1}} W \left|-_{a^{-1}}\right\rangle \left\langle -_{a^{-1}}\right|. \tag{37}$$

with $\alpha_q > 0$ and $\{\left|+_q\right\rangle, \left|-_q\right\rangle\}$ orthonormal bases for $\mathbb{C}^2$.
Combining (35) with (34), one then obtains:

$$\begin{aligned} A_{i^{-1}} {A_i}^\dagger A_{j^{-1}} &= 0, \\ A_i {A_{i^{-1}}}^\dagger A_j &= 0, \\ A_j {A_{i^{-1}}}^\dagger A_i &= 0, \\ A_{j^{-1}} {A_i}^\dagger A_{i^{-1}} &= 0. \end{aligned}$$

By virtue of the same reasoning carried on in A.1 and inserting the A_q, the last equations are equivalent to

$$\begin{aligned}
\langle +_a | +_{a^{-1}} \rangle \langle -_{a^{-1}} | W^\dagger V | +_a \rangle &= 0, \\
\langle -_a | -_{a^{-1}} \rangle \langle +_{a^{-1}} | W^\dagger V | -_a \rangle &= 0, \\
\langle +_a | -_{a^{-1}} \rangle \langle +_{a^{-1}} | W^\dagger V | +_a \rangle &= 0, \\
\langle -_a | +_{a^{-1}} \rangle \langle -_{a^{-1}} | W^\dagger V | -_a \rangle &= 0.
\end{aligned} \tag{38}$$

Accordingly, the possible cases are two, similarly to A.1, namely:

I

$$\begin{aligned}
A_a &= \alpha_a \mu W |+\rangle \langle +| \\
A_b &= \alpha_b \nu W |-\rangle \langle -| \\
A_{a^{-1}} &= \alpha_{a^{-1}} W |+\rangle \langle +| \\
A_{b^{-1}} &= \alpha_{b^{-1}} W |-\rangle \langle -|
\end{aligned}$$

II

$$\begin{aligned}
A_a &= \alpha_a \mu W |+\rangle \langle +| \\
A_b &= \alpha_b \nu W |-\rangle \langle -| \\
A_{a^{-1}} &= \alpha_{a^{-1}} W |-\rangle \langle -| \\
A_{b^{-1}} &= \alpha_{b^{-1}} W |+\rangle \langle +|
\end{aligned}$$

with μ, ν phase factors and each of them can be equal either to i or to $-i$. The two cases are connected through the swap $a^{-1} \leftrightarrow b^{-1}$.
Finding W and imposing the normalization and isotropy condition as done in A.1, one finally finds the A^{I}_q: defining $\zeta^\pm := \frac{\pm i+1}{2}$, the transition matrices are

$$A^{I}_a = \zeta^\pm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^{I}_b = \zeta^\pm \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$A^{I}_{a^{-1}} = \zeta^\mp \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv A^{I}_a{}^\dagger, \qquad A^{I}_{b^{-1}} = \zeta^\mp \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \equiv A^{I}_b{}^\dagger.$$

Remark A.3. *Case I and case II are connected by the swap $a^{-1} \leftrightarrow b^{-1}$; however, it exists neither a unitary nor an antiunitary which connects the two cases: transposition*

affects just $\zeta^{\pm}$*, while a unitary matrix which takes* $A_{a^{-1}}$ *in* $A_{b^{-1}}$ *and the other way around would necessarily swap also* A_a *and* A_b*.*
It is worth noting that an antiunitary transformation does not change the unitarity conditions (and in this sense two sets can be regarded as equivalent, uniquely determined by the 3), but viceversa *it might be false in general that these are in one-to-one correspondence with the relators of a given presentation.*

Remark A.4. *The unitary matrix which represents the action of the isotropy group of* C *is* σ_x*.*
The sets of matrices found are actually four, since in both the main cases the multiplicative constant can assume two values: in section 6.1 it is provided an explanation of this fact.
Furthermore, one has to take into account that by left multiplication of a unitary matrix which commutes with σ_x*—whose general form is*

$$U_{\pm} = a\mathbb{1} \pm ib\sigma_x \quad \textit{for } a, b \geqslant 0 \textit{ and } a^2 + b^2 = 1\text{—}$$

the unitarity conditions do not change, so cases I and II define an entire class of transition matrices for C*.*

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