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DYNAMICAL COMPUTATIONAL NETWORKS
(Reti dinamiche per la computazione)

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LA TERRA

~~ÄST~~ A' SETE

DI PIOGGIA

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Alla mia bisnonna Amelia
ai miei nonni Mario, Gigi, Anna, Bambina
a mia zia Piera
con gratitudine e riconoscenza

Contents

Introduction

The original results we achieved in this thesis – which are collected in Chapter 3 – consist in the realisation of two quantum protocols in order to program dispositions or permutations of N unitary quantum channels according to the state of a control register and resorting in the first case to an ordinary quantum circuit [3, 29] and then to a dynamic computational network, namely a quantum network in which one is allowed to use a particular oracle that is *a priori* admissible in quantum mechanics and yet unfeasible through the quantum computing model devised by Deutsch: the *Quantum Switch Oracle* [6, 31].

In particular, we will firstly conceive an ordinary quantum circuit which is able to program all the possible dispositions of N unitary channels and superpositions of them according to the state of a control register in the most efficient way, estimating its complexity. Then it will be considered the possibility of programming all the possible permutations of N unitary channels and superpositions of them according to the state of a control register through a dynamic computational network. The required computational resources will be estimated in both cases and we will prove that one can efficiently accomplish the second task resorting to a single call of each channel but paying the price of a little greater control register in respect to the first task. Nevertheless the number of the required elementary operations is of the same order of magnitude in both cases.

The topics we will deal with in the first two chapters are instead preparatory to the comprehension of the third one. In Chapter 1 we will sum up some results of ordinary quantum computation, giving an account of elementary operations in quantum circuits and identifying the subset of higher-order maps which are both admissible according to quantum mechanics and feasible through Deutsch's quantum computers [8, 32, 28, 29].

In the second chapter we will analyse those maps that cannot be realised through an ordinary quantum circuit, even if they are admissible according to quantum mechanics. A capital example of them is the aforementioned *Quantum Switch Oracle* [6, 31].

Finally, some useful mathematical and notational prerequisites following on from Refs. [29, 31] and necessary for the comprehension of this work are reported in Appendix A.

Introduzione

Le parti originali del presente lavoro di tesi, raccolte nel terzo capitolo, sono volte alla realizzazione di due protocolli quantistici con cui programmare disposizioni o permutazioni di N canali unitari in modo controllato, ricorrendo in un primo momento ad un *circuito quantistico ordinario* [3, 29] ed in seguito ad una *rete computazionale dinamica,* ovvero una rete quantistica in cui è consentito l'utilizzo di un oracolo particolare, ammissibile in meccanica quantistica, ma non realizzabile mediante il modello di computer quantistico concepito da Deutsch: il *Quantum Switch Oracle* [6, 31].

Progetteremo anzitutto un circuito quantistico in grado di programmare in modo controllato tutte le possibili disposizioni di N canali unitari e loro sovrapposizioni nel modo più efficiente consentito dalla computazione quantistica ordinaria, valutandone la complessità. Verrà poi considerata la possibilità di programmare tutte le possibili permutazioni controllate di N canali e loro sovrapposizioni mediante una rete computazionale dinamica. In particolare si dimostrerà che ciò è realizzabile in maniera efficiente ricorrendo ad una singola chiamata dei canali in input, ma richiedendo un registro di controllo con un numero di qubit ancillari leggermente maggiore rispetto alla situazione precedente. Tuttavia, il numero di operazioni elementari richieste è dello stesso ordine di grandezza in entrambi i casi.

Nei primi due capitoli vengono invece trattati argomenti propedeutici alla comprensione del terzo. Nel primo si riassumono alcuni risultati di computazione quantistica ordinaria, riportandone le operazioni elementari ed individuando il sottoinsieme di mappe di ordine superiore che siano allo stesso tempo ammissibili secondo la meccanica quantistica e realizzabili mediante il modello teorico di computer quantistico proposto da Deutsch [8, 32, 28, 29].

Nel secondo capitolo si analizzeranno quelle mappe che, pur essendo ammissibili secondo la meccanica quantistica, non possono essere realizzate implementando un circuito quantistico ordinario, tra quali rientra anche il *Quantum Switch Oracle* [6, 31].

Da ultimo, in Appendice A sono riportati alcuni prerequisiti matematici e notazionali stabiliti nelle referenze [29, 31] necessari per la comprensione del presente lavoro.

Chapter 1

Achievable transformations through ordinary quantum circuits

1.1 The Quantum Circuit Model

Quantum computation and quantum information theory are research fields that deal with the study of the information processing tasks that can be accomplished using quantum mechanical resources.

In the early 80's two works were published independently by Manin [1] and by Feynman [2], who conjectured that quantum computers could simulate quantum systems in a more efficient way than classical computers. The first model of quantum computation was established in 1985 by Deutsch [3]. Deutsch tried to define a computational device that would be capable of effectually simulating an arbitrary quantum computational process and named it *Quantum Turing Machine*, in analogy with the *Classical Turing Machine* introduced by Turing in 1936 [4].

Unfortunately Quantum Turing Machines are not simple devices to deal with. Indeed the analysis of Quantum Turing Machines is complicated by the fact that not only the data, but also the control variables (*e.g.* head position) can exist in a superposition of classical states. To compensate this inconvenience, Deutsch himself conceived in 1989 the *Quantum Circuit Model,* which is the natural quantum generalization of acyclic combinational logic circuits studied in conventional computational complexity theory [5]. Quantum circuits are built up by *quantum gates*, interconnected by *quantum wires* complying with the following

Prescriptions (Quantum Circuit Model)

1. Qubits are represented by wires;

2. A box on a single wire represents a transformation on the corresponding system, whence a box on multiple wires generally describes an interaction between the corresponding systems;

3. Input/output relations proceed from left to right and there are no loops in the circuit;

4. Each box represents a single use of the corresponding transformation.

In order to have reversible quantum computation, each gate has the same number of inputs as outputs and a gate with n inputs performs an unitary operation belonging to the group of the rotations in a 2^n-dimensional space $\mathrm{U}(2^n)$. Instead each wire represents a quantum system associated to a 2-dimensional Hilbert space, namely a *qubit*.

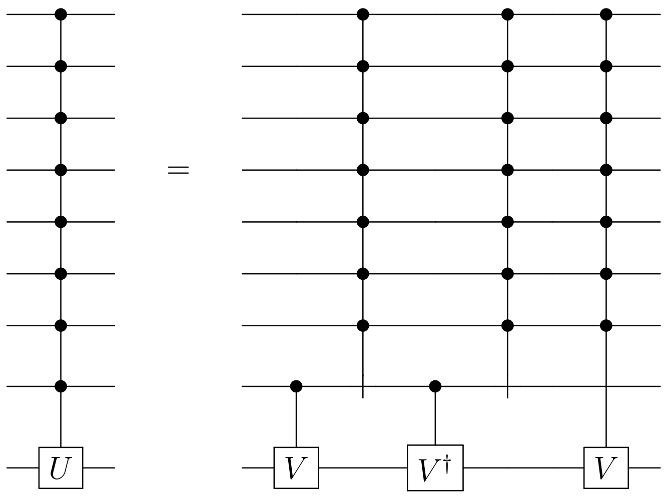
Describing a computational process with a Quantum Turing Machine or with a quantum circuit was proven by Yao to be equivalent for a class of computation that can be described as processing of input qubits [7]. Working with a quantum circuit is definitely easier than dealing with a Quantum Turing Machine, since the control variables (*e.g.* the wiring diagram) can be thought as classical while the data in the wires and the logic gates are obviously quantum. One can also better evaluate the amount of computational resources required by any algorithm (*e.g.* number of oracle calls, number of qubits, length of the computation, etc.).

As Ref. [6] suggests, it is worth stressing that a quantum circuit is a computational circuit and not a physical one: While in a physical circuit we can have loops (for example when a system passes twice through the same physical device), in a computational circuit there are no loops (when we apply twice a transformation to the same system we just draw two times the same box). A computational circuit represents the actual flow of information during the run of a program. It is also important to make clear the distinction between *program* and *computational circuit,* the former being a set of instructions to build up the latter. In a computational circuit, wires can never go backward, because this would mean to go backward in time, whereas, on the contrary, in its program code we can have commands pointing back to previous instructions.

There are many more choices for the set of universal gate(s) in reversible quantum computation than in its classical analogue, as remembered in Ref. [8]. DiVincenzo [9] proved that two-qubits universal quantum gates are possible; Barenco [10] extended this to show that almost any two-qubits gate (within a certain restricted class) is universal; Lloyd and Deutsch *et al.* independently showed that almost any two-qubits or n-qubits ($n \geqslant 2$) quantum logic gate is also universal; Barenco *et al.* showed that a non universal classical two-qubits gate (the `CNOT` gate) in conjunction with quantum single qubit gates is also universal, also exhibiting a number of efficient schemes aimed to build up certain classes of n-qubit operations only employing the set of gates they proved to be universal.

Regarding controlled operations, we want to report three fundamental results contained in Ref. [8] due to their importance:

Lemma 1.1.1. *For any unitary* 2×2 *matrix* U*, a* $C^{(n-1)}(U)$ *gate can be simulated by a network of the form:*



(1.1)

(illustrated for $n = 9$*), where* V *is unitary.*

Corollary 1.1.1. *For any unitary* U*, a* $C^{(n-1)}(U)$ *gate can be simulated in terms of* $\Theta(n^2)$ *basic operations.*

Lemma 1.1.2. *Any simulation of a nonscalar* $C^{(n-1)}(U)$ *gate (i.e. where* $U \neq \mathrm{Ph}(\delta) \cdot I$*) requires at least* $n - 1$ *basic operations.*

The quantum circuit framework has been a fertile ground for the development of quantum algorithms. Over the years many situations have been detected where quantum computers reveal themselves more powerful than the classical ones [3, 11, 12, 13, 14]. Roughly speaking, this can be understood via the following argument [8]: While reversible classical computation is contained within quantum mechanics, the first is only a small subset of the latter, since the time evolution of a classical reversible computer is described by unitary operators whose matrix elements are only zero or one, arbitrary complex numbers being not allowed. Unitary time evolution can of course be simulated by a classical computer (*e.g.* an analog optical computer governed by Maxwell's equations)[15], but the dimension of the thus attainable unitary evolution operator is bounded by the number of classical degrees of freedom – *i.e.* roughly proportional to the size of the apparatus. By contrast a quantum computer with m physical bits can perform unitary operations in a 2^m-dimensional space, which is exponentially larger than its physical size. Moreover, unlike the classical situation, quantum mechanics allows one to program the state of a qubit in a superposition of two Boolean states and to entangle the qubit with the states of other qubits.

Some quantum algorithms were found accomplishing their task requiring a polynomial number of elementary operations, unlike the more complex classical analogous ones. This is *e.g.* the case of the celebrated *factoring* and *discrete log algorithms* by Shor [16]. There also exist other algorithms that solve a specific problem with fewer computational steps in respect to the classical case, although they belong to the same complexity class, as it happens for *Grover's search algorithm* [17, 18].

1.2 From quantum channels to higher-order quantum maps

States, POVMs and quantum operations constitute the elementary objects of any ordinary quantum circuit. The development of the mathematical formalism of quantum mechanics led to a deeper understanding of which physical transformations are *in principle* admissible according to the theory itself. In particular, a complete characterisation of the transformations involving quantum systems has been stated in terms of linearity, complete positivity and (in the deterministic case) trace preservation [19, 20, 21, 22, 23, 24]. Overall, this research field has clarified that the essence of quantum mechanics lies in its probabilistic structure and that the mathematical constraints that guarantee the admissibility of a quantum map are exactly those required in order to allow a consistent probabilistic interpretation.

Channels and POVMs provide an efficient description of elementary circuits that transform or measure quantum states. One can obviously combine elementary circuits in a larger quantum network[1], taking care of the aforementioned rules requested by the Quantum Circuit Model, thus broadening the variety of possible tasks that one can perform. For example, quantum computing networks can be used as programmable machines, which implement different transformations on input data depending on the quantum state of the program. In some cases the program itself can be a quantum channel, rather than a state: During computation the network could call a variable channel as a subroutine, so that the overall transformation of the input data is programmed by it. Even more generally, the action

[1] This kind of networks should not be confused with *Dynamical Computational Networks,* which will be introduced in Chapter 2.

of the network can be programmed by a sequence of variable states and channels that are called at different times, that is, at different steps of the computation. A similar situation arises in multiple-rounds quantum games [26], where the overall outcome of the game is determined by the sequence of moves (state-preparations, measurements, and channels) performed by different players. For example, in a two-party game Alice's strategy can be seen as a particular quantum network in which Bob's moves act as variable subroutines. Of course, the subroutines corresponding to Bob's moves are in turn parts of Bob's network, so that the whole protocol can be seen as the interlinking of two networks corresponding to Alice's and Bob's strategies.

What we just explained reveals that quantum networks can be definitely used in a number of different ways, each of them corresponding to a different kind of transformation achievable with them, *e.g.* transformations from states to channels, from channels to channels, and from sequences of states or channels to channels. Luckily enough, an efficient treatment of quantum networks is possible, despite the infinity of different transformations associated to them if one tackles a mathematical characterisation of the so called *higher-order quantum maps,* namely maps that transform other maps compatibly with the structure of quantum mechanics.

In the following sections of this chapter, we will discuss two equivalent approaches intended to characterise quantum maps of every order. The first is a constructive one: It is the one sketched by Kretschmann and Werner to encompass causal automata [25] and later exploited by Gutoski and Watrous to establish a general theory of quantum games [26]. A clear satisfying treatment thereof can be also found in Ref. [29] by Chiribella, D'Ariano and Perinotti. The second one makes its moves from an axiomatical point of view, providing the definition of *quantum combs* satisfying only one causal structure and quantum maps with a definite causal structure thereof [27, 28, 29]. Further investigations – which will be treated in Chapter 2 – tackled the problem of characterising combs satisfying more than one causal structure and admissible quantum maps thereon [30, 31]. As we will see, the feasibility of the latter manifests some open problems up to now.

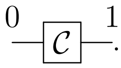
1.3 Constructive approach to quantum maps

We devote this section to an *excursus* about the fundamental objects characterising the constructive approach of quantum networks, also discussing their mathematical properties, for which a physical interpretation in provided in Refs. [26, 28, 29, 39]. In the following, we will extensively use the notation introduced in Appendix A.

1.3.1 Deterministic Choi-Jamiołkowski operators

In the general description of quantum mechanics, *quantum states* are described through density matrices on Hilbert space $\mathcal{H}$ of the system, *i.e.* positive semidefinite operators $\rho \in \mathcal{L}(\mathcal{H})$ with $\mathrm{Tr}[\rho] = 1$.

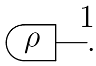
Deterministic transformations of quantum states are the so-called *quantum channels,* a quantum channel $\mathcal{C}$ from states on $\mathcal{H}_0$ to states on $\mathcal{H}_1$ being a trace-preserving completely positive map, with diagrammatic representation



(1.2)

According to Lemmas A.1.1, A.1.2, A.1.3, the Choi-Jamiołkowski operator corresponding to $\mathcal{C}$ is a positive semidefinite operator $C \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ satisfying $\mathrm{Tr}_{\mathcal{H}_1}[C] = I_{\mathcal{H}_0}$.

It is immediate to see that a density matrix is a particular case of Choi-Jamiołkowski operator of a channel with one-dimensional input space $\mathcal{H}_0$: in this case the condition $\mathrm{Tr}_{\mathcal{H}_1}[C] = I_{\mathcal{H}_0}$ becomes indeed $\mathrm{Tr}[C] = 1$. This reflects the fact that holding a quantum state is equivalent to having at one's disposal one use of a suitable preparation device. Thus a state is represented by

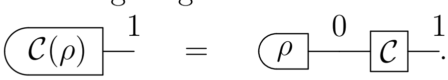


(1.3)

The application of a channel $\mathcal{C}$ to a state ρ is equivalent to the composition of two channels, and is indeed given by the link product of the corresponding Choi-Jamiołkowski operators

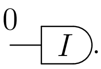
$$\mathcal{C}(\rho) = C * \rho, \tag{1.4}$$

which agrees both with Eq. (A.9) and Theorem A.2.1. This situation can be represented through the following diagram:



(1.5)

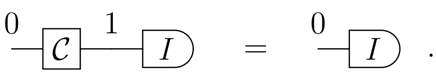
The opposite example is the completely demolishing *trace channel* $\mathcal{T}(\rho) = \mathrm{Tr}[\rho]$, which transforms quantum states into their probabilities (of course, normalised density matrices give unit probabilities): This channel has one-dimensional output space $\mathcal{H}_1$, and, accordingly its Choi-Jamiołkowski operator is $T = I_{\mathcal{H}_0}$. We picture this channel as



(1.6)

Notice that the normalisation condition of the Choi-Jamiołkowski operator $C \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ of a channel $\mathcal{C}$ can be also written in terms of concatenation with the trace channel as

$$C * I_{\mathcal{H}_1} = I_{\mathcal{H}_0}$$



(1.7)

1.3.2 Probabilistic Choi-Jamiołkowski operators

In addition to the Choi-Jamiołkowski operators of deterministic quantum devices, with which we have dealt in the previous subsection, one can consider their probabilistic versions. A complete family of probabilistic transformations from states on $\mathcal{H}_0$ to states on $\mathcal{H}_1$, known as *quantum instrument*, is a set of CP maps $\{\mathcal{C}_i \mid i \in I\}$ summing up to a trace-preserving CP map $\mathcal{C} = \sum_{i\in I} \mathcal{C}_i$. The corresponding Choi-Jamiołkowski operators $\{C_i \mid i \in I\}$ are positive semidefinite operators summing up to a deterministic Choi-Jamiołkowski operator $C = \sum_{i\in I} C_i$ with $C * I_{\mathcal{H}_1} = I_{\mathcal{H}_0}$. For families of probabilistic transformations, the index i has always to be intended as a classical outcome, that is known to the experimenter, and heralds the occurrence of different random transformations.

For one-dimensional input space $\mathcal{H}_0$, a complete family of probabilistic Choi-Jamiołkowski operators $\{\rho_i \mid i \in I\}$ with $\sum_i \rho_i = \rho, \mathrm{Tr}[\rho] = 1$ describes a *random source of quantum states*. Applying the trace channel $\mathcal{T}$ after the source gives the probability of the source emitting the i-th state: $p_i = \mathrm{Tr}[\rho_i] = \rho_i * I_{\mathcal{H}_1}$ (of course $p_i \geq 0$ and $\sum_i p_i = 1$).

For one-dimensional output space $\mathcal{H}_1$, a complete family of probabilistic Choi-Jamiołkowski operators is instead a POVM $\{P_i \mid i \in I\}$, $\sum_i P_i = I_{\mathcal{H}_1}$.

Measuring the POVM on a state ρ is equivalent to applying the random device described by $\{P_i\}$ after the preparation device for the state ρ, producing as outcomes the probabilities

$$p(i|\rho) = \rho * P_i = \mathrm{Tr}[\rho P_i^T]. \tag{1.8}$$

Apart from the transpose, which can be absorbed in the definition of the POVM, this is nothing but the Born rule for probabilities, obtained here from the composition of a preparation channel with a random transformation with one-dimensional output space.

In conclusion, states, channels, random sources of quantum states, quantum instruments, and POVMs can be treated on the same footing as deterministic and probabilistic transformations, which in turn can be described using only Choi-Jamiołkowski operators and link product.

1.3.3 Memory channels

As it was said in Sec. 1.2, an ordinary quantum network is obtained by properly assembling elementary circuits, each of them represented by its Choi-Jamiołkowski operator. One can adopt the following convention, which appears to be very convenient for the description of quantum networks: If an elementary circuit is run more than once, *i.e.* at different steps of the computation, one has to attach a different label to each different use, so that different uses of the same circuit are actually considered as different circuits.

To build up a particular quantum network one needs in principle to have at disposal the whole list of elementary circuits and a list of instructions about how to connect them. In connecting circuits there are clearly two restrictions:

i) One can only connect the output of a circuit with the input of another circuit;

ii) There cannot be cycles.

These restrictions raise from the rules stated for the Quantum Circuit Model, which we have reported in Sec. 1.1. In particular, they ensure causality, namely the fact that quantum information in a network flows from input to output without loops. This implies that the connections in a quantum network can be represented with a directed acyclic graph (DAG), where each vertex represents a quantum circuit, and each arrow represents a quantum system travelling from one circuit to another, as it is shown *e.g.* in Fig. 1.1.

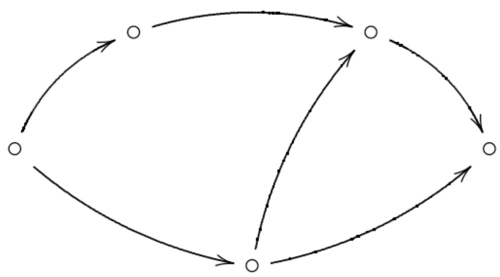


Figure 1.1

Notice that such a graph represents only the internal connections of a network, while to have a complete graphical representation one should also append to the vertices a number of free incoming and outgoing arrows correspondingly to quantum systems that enter or exit the network. In other words, the graphical representation of a quantum network is provided by a DAG where some sources (vertices without incoming arrows) and some sinks (vertices without outgoing arrows) have been removed. Fig. 1.2 stands for an example of this situation. The free arrows remaining after having removed a source represent input systems entering the network, while the free arrows remaining after removing a sink represent output systems exiting the network.

The flow of quantum information along the arrows of the graph induces a partial ordering of the vertices: We say that the circuit in vertex v_1 *causally precedes* the

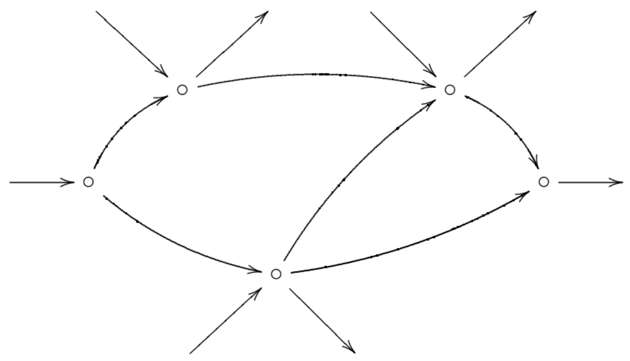


Figure 1.2

circuit in vertex v_2 ($v_1 \preceq v_2$) if there is a directed path from v_1 to v_2. A well known theorem in graph theory states that for a DAG there always exists a way to extend the partial ordering $\preceq$ to a total ordering $\leq$ of the vertices. Intuitively speaking, the relation $\leq$ fixes a schedule for the order in which the circuits in the network can be run, compatibly with the causal ordering of input-output relations. In general, the total ordering $\leq$ is not uniquely determined by the partial ordering $\preceq$: The same quantum network can be used in different ways, corresponding to different orders in which the elementary circuits are run. For instance, Fig. 1.3 reproduces a totally ordered quantum network obtained by arranging from left to right the vertices of the diagram in Fig. 1.2 according to a sequential ordering compatible with the causal ordering fixed by input-output relations.

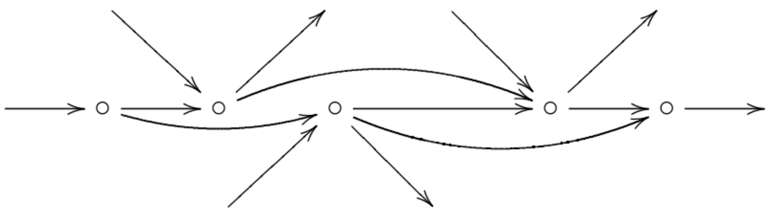


Figure 1.3

An ordinary quantum network with a given sequential ordering of the vertices becomes an ordinary compound quantum circuit, in which different operations are performed according to a precise schedule. Totally ordered quantum networks have a large number of applications in quantum information, and, accordingly, they have been given different names, depending on the context. For example, they are referred to as *quantum strategies* in quantum game theoretical and cryptographic applications [26]. Moreover, a totally ordered quantum network is equivalent to a sequence of channels with memory, as illustrated in Fig. 1.4 [28].

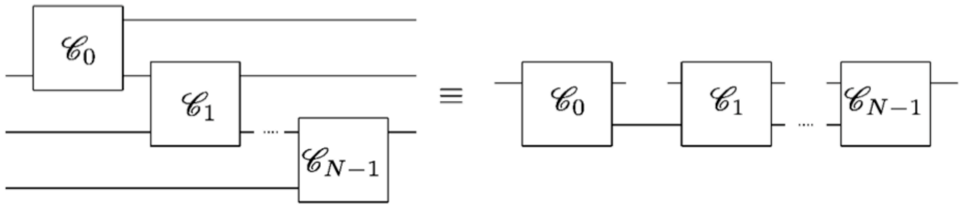


Figure 1.4

We now report some results about deterministic quantum networks. We omit their generalisation to ordinary probabilistic quantum networks, which is not difficult to obtain. First of all, exploiting the associatively property of the link product given by Theorem A.2.2, it is straightforward to prove that the Choi-Jamiołkowski operator describing a deterministic network made up by N deterministic elementary circuits described by the Choi-Jamiołkowski operators $\{C_0, \ldots, C_{N-1} \,|\, C_0 \leqslant \ldots \leqslant C_{N-1}\}$ is given by

$$R^{(N)} = C_0 * C_1 * \cdots * C_{N-1} = \mathop{\ast}_{j=0}^{N-1} C_j. \tag{1.9}$$

Furthermore the following theorem applies to deterministic memory channels, providing a characterisation of their Choi-Jamiołkowski operators [29]:

Theorem 1.3.1. *Let* $R^{(N)} \in \mathcal{L}(\bigotimes_{j=0}^{2N-1} \mathcal{H}_j)$ *be a positive operator satisfying the relations*

$$\begin{aligned} \mathrm{Tr}_{2j-1}[R^{(j)}] &= I_{2j-2} \otimes R^{(j-1)}, \quad 2 \leqslant j \leqslant N \\ \mathrm{Tr}_1[R^{(1)}] &= I_0. \end{aligned} \tag{1.10}$$

where $R^{(j)}, 1 \leq j \leq n-1$ *are suitable positive operators. Then* $R^{(N)}$ *is the Choi-Jamiołkowski operator of a memory channel.*

Eq. (1.10) is not a merely mathematical property but it has an important physical interpretation. In fact it implies that the output of circuit $\mathcal{C}_i$ is independent of the output of circuit $\mathcal{C}_j$ whenever $\mathcal{C}_i$ causally precedes $\mathcal{C}_j$, namely whenever $\mathcal{C}_i \leqslant \mathcal{C}_j$. This reflects the fact that the causal structure of such memory channels is fixed, thus compelling *the flow of information* to proceed from the input of the first channel to the output of the last channel without temporal loops. Theorem 1.3.1 categorically assures us that Rule 3 characterising the Quantum Circuit Model is not violated in setting up the just outlined constructive approach.

1.4 Axiomatic approach to quantum maps

An alternative procedure to the just treated constructive one can be set up through an equivalent axiomatic approach, as it is pointed out in Refs. [27, 28, 29] and then revised in Ref. [31]. This approach has at its root the investigation of the requirements that a quantum maps must fulfil *in principle* if one wants to preserve the probabilistic interpretation of quantum mechanics. Taking on this task, one should contemporarily discuss the type-theoretical aspects of higher-order quantum maps and their natural domain definition.

We begin our discussion remembering that the most general transformations of quantum states that can be performed in quantum mechanics are quantum operations, the deterministic subset of which is the set of quantum channels. Quantum operations satisfy the two minimal requirements of linearity and complete positivity[2].

Linearity is required by the probabilistic structure of quantum mechanics. Indeed, if we apply the transformation $\mathcal{C}$ to the state $\rho = \sum_i p_i \rho_i$ – corresponding to a random choice of the states $\{\rho_i\}$ with probabilities $\{p_i\}$ – then the output state must be a random choice of the states $\{\mathcal{C}(\rho_i)\}$ with the same probabilities, *i.e.* $\mathcal{C}(\rho) = \sum_i p_i \mathcal{C}(\rho_i)$. For the same reason, we should also have $\mathcal{C}(p\rho) = p\,\mathcal{C}(\rho)$ for any $0 \leq p \leq 1$. These two conditions together imply that $\mathcal{C}$ can be extended without loss of generality to a linear map on $\mathcal{L}(\mathcal{H}_S)$, $\mathcal{H}_S$ being the Hilbert space of the system.

Complete positivity is supposed to hold if we want the transformation $\mathcal{C}$ to produce a legitimate output $\mathcal{C} \otimes \mathcal{I}_A(\rho_{SA})$ when acting locally on a bipartite input state ρ_{SA} on $\mathcal{H}_S \otimes \mathcal{H}_A$: in this case, this means that we want the output $\mathcal{C} \otimes \mathcal{I}_A(\rho_{SA})$ to be a positive matrix for any positive input ρ_{SA}.

Quantum channels are also trace preserving since they are deterministic due to their definition, namely if $\mathcal{C}$ is a quantum channel the equality $\mathrm{Tr}\,[\rho] = \mathrm{Tr}\,[\mathcal{C}(\rho)]$ must hold for all input states ρ.

Let us raise the level from states to channels, and ask what are the admissible transformations of channels. Let us consider maps $\tilde{\mathcal{S}}$ from linear maps $\mathcal{T} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ to linear maps $\mathcal{T}' : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_3)$.

In order to have compatibility with the probabilistic structure of quantum mechanics, two conditions are required on the map $\tilde{\mathcal{S}}$: It must be linear and it

[2] See *e.g.* [32].

must preserve complete positivity with respect to any extension. Thus we can state the following [27]:

Admissibility conditions for $\tilde{\mathcal{S}}$

i) Linearity;

ii) Local Complete Positivity, *i.e.* $\tilde{\mathcal{S}}$ it is supposed to preserve complete positivity with respect to any extension.

The property of being locally CP will be formally given in Def. 2.2.5. In practice, condition *ii)* requires that $\tilde{\mathcal{S}}$ must preserve complete positivity, also when applied locally on some bipartite map. More explicitly, we require that if

$$\mathcal{R} : \mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_2) \otimes \mathcal{L}(\mathcal{H}_B) \tag{1.11}$$

is CP, then also

$$\mathcal{R}' \doteq (\tilde{\mathcal{S}} \otimes \tilde{\mathcal{I}})(\mathcal{R}) : \mathcal{L}(\mathcal{H}_0) \otimes \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_3) \otimes \mathcal{L}(\mathcal{H}_B) \tag{1.12}$$

is CP. We will come back on this tricky point in Sec. 2.2.2, in which it will be also proved that a locally CP map is also a CP map [31].

An equivalent characterisation of these conditions can be obtained considering the conjugate map $\mathcal{S}$ of $\tilde{\mathcal{S}}$, defined as follows:

$$\mathcal{S} \doteq \mathfrak{C} \circ \tilde{\mathcal{S}} \circ \mathfrak{C}^{-1}, \tag{1.13}$$

which transforms the Choi-Jamiołkowski operator T of $\mathcal{T}$ into the Choi-Jamiołkowski operator T' of the map $\mathcal{T}'$ [3]. The constraints that $\mathcal{S}$ must satisfy to be declared admissible are analogous to the ones we stated for $\tilde{\mathcal{S}}$ [27, 29]. Since $\mathcal{S}$ is in one-to-one correspondence with $\tilde{\mathcal{S}}$, we associate the Choi-Jamiołkowski operator S of $\mathcal{S}$ to both of them. In this work we will systematically work with the map $\mathcal{S}$ instead of $\tilde{\mathcal{S}}$ for simplicity, however the whole construction that follows must be intended as dealing with transformations of transformations rather than with transformations of operators, thus generating an infinite hierarchy of higher-rank quantum maps.

We now individuate a special class of higher-order quantum maps introducing the following

Definition 1.4.1. *A* quantum 1-comb *on* $(\mathcal{H}_0, \mathcal{H}_1)$ *is the Choi-Jamiołkowski operator of a linear CP map* $\mathcal{S}^{(1)} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_1)$*, namely*

$$S \doteq \mathfrak{C}(\mathcal{S}^{(1)}). \tag{1.14}$$

[3] The map $\mathfrak{C}$ is formally defined in Def. A.1.1.

For $N \geq 2$ *a* quantum N-comb *on* $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ *is the Choi-Jamiołkowski operator of a linear CP N-map, i.e. a linear CP map transforming* $(N-1)$*-combs on* $(\mathcal{H}_1, \ldots, \mathcal{H}_{2N-2})$ *into 1-combs on* $(\mathcal{H}_0, \mathcal{H}_{2N-1})$*.*

The set of generic N-combs defined on $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ *will be denoted with* $\mathrm{comb}\,(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$*.*

Quantum combs can be classified whether they are deterministic or probabilistic:

Definition 1.4.2. *A* deterministic *1-comb is the Choi-Jamiołkowski operator of a channel. A* deterministic *N-comb* $S^{(N)}$ *is the Choi-Jamiołkowski operator of a* deterministic *N-map, i.e. a map* $\mathcal{S}^{(N)}$ *that transforms deterministic* $(N-1)$*-combs into deterministic 1-combs.*

Definition 1.4.3. *An N-comb* $R^{(N)}$ *on* $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ *is* probabilistic *if there is a deterministic N-comb* $S^{(N)}$ *such that* $R^{(N)} \leq S^{(N)}$*.*

From a type-theoretic point of view Def. 1.4.1 generates an infinite hierarchy of types via the following procedure:

- A quantum 1-comb is the type of a quantum channel;
- A quantum $(M+1)$-comb is defined as a map from M-combs to quantum channels.

These types are called *N-maps.* As we will see in Chapter 2, N-maps do not cover all the admissible maps one can define in quantum mechanics.

We now state some theorems that outline the set of quantum N-combs providing some properties of them [29]. The first one is a capital characterisation theorem:

Theorem 1.4.1. *A positive operator* $S^{(N)}$ *on* $\bigotimes_{k=0}^{2N-1} \mathcal{H}_k$ *is a deterministic N-comb if and only if the following identity holds:*

$$\begin{aligned} \mathrm{Tr}_{2j-1}[S^{(j)}] &= I_{2j-2} \otimes S^{(j-1)}, \quad 2 \leqslant j \leqslant N \\ \mathrm{Tr}_1[S^{(1)}] &= I_0, \end{aligned} \tag{1.15}$$

where $\{S^{(j)} \,|\, j = 1, \ldots, N-1\}$ *are deterministic j-combs. Equivalently:*

$$\begin{aligned} S^{(j)} * I_{2j-1} &= S^{(j-1)} * I_{2j-2}\,, \quad 2 \leqslant j \leqslant N \\ S^{(1)} * I_1 &= I_0. \end{aligned} \tag{1.16}$$

In the proof of Theorem 1.4.1 the following Lemma plays a crucial role:

Lemma 1.4.1. *The set of positive operators* $R^{(N)}$ *such that* $R^{(N)} \leqslant S^{(N)}$ *for some* $S^{(N)}$ *satisfying Eq.* (1.15) *generates the positive cone in* $\mathcal{L}(\bigotimes_{k=0}^{2N-1} \mathcal{H}_k)$.

Lemma 1.4.1 is also important since it shows that the cone generated by probabilistic N-combs in $\mathcal{L}(\bigotimes_{k=0}^{2N-1} \mathcal{H}_k)$ is the whole cone of positive operators. Thus Theorem 1.4.1 proves that the deterministic N-combs $S^{(N)} \in \mathrm{comb}(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ form a convex set K_N which is the intersection of the cone of positive operators with the hyperplanes defined by Eq. (1.15), which is the mathematical translation of causal ordering. Indeed this equation reflects the semicausality property [34] for transformations occurring at teeth j and i, with $j < i$, namely the independence of the j-th transformation from the i-th transformation for $j < i$. In other words a system j can transmit information only to that systems i such that $i > j$, while other possibilities are excluded.

From the comparison of Theorem 1.3.1 with Theorem 1.4.1 we straightforwardly derive the following equivalence theorem:

Corollary 1.4.1. *A deterministic* N*-comb is also the Choi-Jamiołkowski operator of an* N*-partite memory channel.*

A realisation scheme for any admissible N-map is provided by the following

Theorem 1.4.2 (Realisation of admissible N-maps)**.** *For all* N*, any deterministic* N*-map* $\tilde{\mathcal{S}}^{(N)}$ *can be achieved by a physical scheme corresponding to the memory channel whose deterministic* N*-comb is* $S^{(N)}$*. Let* $T^{(N-1)}$ *be any* $(N-1)$*-comb in* $\mathrm{comb}(\mathcal{H}_1, \ldots, \mathcal{H}_{2N-2})$*. The transformation*

$$\tilde{\mathcal{S}}^{(N)} : \tilde{\mathcal{T}}^{(N-1)} \mapsto \tilde{\mathcal{T}}'^{(1)} = \tilde{\mathcal{S}}^{(N)} \left(\tilde{\mathcal{T}}^{(N-1)} \right) \tag{1.17}$$

can be achieved by connecting the two memory channels represented by $S^{(N)}$ *and* $T^{(N-1)}$ *as in Fig. 1.5.*

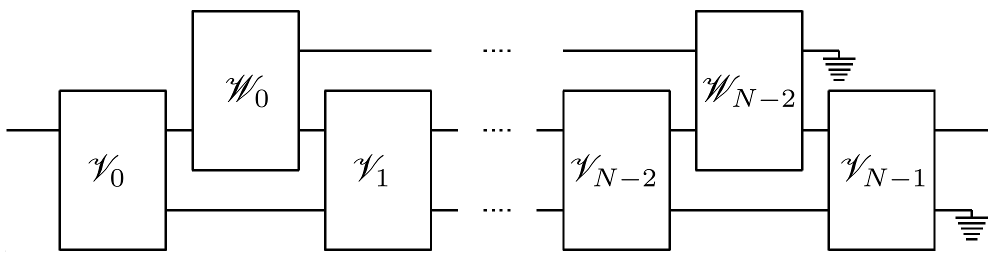


Figure 1.5

It is worth noticing that, within this axiomatic framework, N-partite memory channels are derived from the recursive construction of admissible higher-order maps, rather than being assumed as a particular type of channels with additional causal structure. In the axiomatic approach, the quantum memory channel emerges in the Russian-dolls construction of maps on maps and the causal structure is generated by the map-recursion.

We will see in the following chapter that does exist other maps that are admissible according to quantum theory, but that they cannot be traced back to N-maps, thus being not achievable through implementing any memory channel.

Chapter 2

Admissible $(N \to M)$-maps

In this chapter we will pursue the study of another class of admissible quantum maps, namely those maps which take as input an N-comb and output an M-comb without violating the theoretical structure of probability theory. As we will see in Sections 2.1 and 2.2 the treatment of this argument requires a particular attention. Indeed from Refs. [6, 31] one clearly evinces that some simple maps can be doubtless defined, nevertheless they cannot be realised resorting to Theorem 1.4.2. This fact implies that memory channels are not universal for higher-order computation. A further consequence is the fact that the Quantum Circuit Model would not be the ultimate computational model for higher-order quantum maps.

The prototype of those maps that cannot be simulated through resorting to the procedure given by Theorem 1.4.2 is the *Quantum Switch Oracle* (from now shortened in `QSO`), viz. the extension by linearity on the ancillary qubit of the gate that takes as input a control qubit $|x\rangle$ and two quantum operations $-\boxed{f}-$ and $-\boxed{g}-$ and outputs

$$S\left(|x\rangle, -\boxed{f}-, -\boxed{g}-\right) = \begin{cases} -\boxed{f}-\boxed{g}- & \text{if} \quad |x\rangle = |1\rangle \\ -\boxed{g}-\boxed{f}- & \text{if} \quad |x\rangle = |0\rangle \end{cases}. \tag{2.1}$$

We will investigate the physical grounds of this impossibility in Sec. 2.3 and we will provide a probabilistic simulation of the switch of two channels via post-selected teleportation in Sec. 2.4.

2.1 The Currying Theorem

In Sec. 1.4 we dealt with maps of type $N \to 1$, whose Choi-Jamiołkowski operator is a quantum comb by definition, can be realised through a memory channel via an isomorphism relation. Besides $N \to 1$ maps, one can also consider admissible $(N \to M)$-maps, *i.e.* maps transforming N-combs into M-combs and tackle their type-theoretical characterisation along with a suitable realisation procedure thereof. In concert with what we just declared we give the following

Definition 2.1.1. *An* $(N \to M)$-map $\mathcal{S}^{N\to M}$ *is an admissible map transforming* N*-combs into* M*-combs. We say that* $\mathcal{S}^{N\to M}$ *is* deterministic *if it sends deterministic* N*-combs into* M*-combs.*

In this way we are introducing new types $N \to M$, which generally are not reducible to types $N \to 1$ unless we impose some other constraint, as it was made in Ref. [29]. We will come again on this remarkably point in the following of the section. The admissibility condition of a $\mathcal{S}^{N\to M}$ will be studied in the case $N = M = 1$ in Sec. 2.2.2.

We also give the following

Definition 2.1.2. *The product type* $N \times M$ *is the type of the tensor product operator* $R^{(N)} \otimes T^{(M)}$*, where* $R^{(N)}$ *is an* N*-comb and* $T^{(M)}$ *is an* M*-comb.*

We now report the Currying Theorem which states the following type isomorphism among higher-order quantum maps:

$$N \to (M \to 1) \cong N \times M \to 1. \tag{2.2}$$

Theorem 2.1.1 (Currying)**.** *Let* $\mathcal{S}^{N\to M+1}$ *be a* $(N \to M+1)$*-map. Then* $\mathcal{S}^{N\to M+1}$ *is in one-to-one correspondence with a CP map* $\mathcal{S}^{N\times M\to 1}$ *that transforms tensor product operators* $R^{(N)} \otimes O^{(M)}$ *of* N*- and* M*-combs into* 1*-combs. Moreover* $\mathcal{S}^{N\to M+1}$ *is deterministic if and only if* $\mathcal{S}^{N\times M\to 1}$ *transforms tensor product of deterministic combs into channels.*

Proof. Suppose that $\mathcal{S}^{(N\to M+1)}$ maps an N-comb $R^{(N)}$ into an $(M+1)$-comb $R'^{(M+1)} = \mathcal{S}^{(N\to M+1)}(R^{(N)})$. In term of Choi-Jamiołkowski operators we have

$$R'^{(M+1)} = \mathfrak{C}(\mathcal{S}^{(N\to M+1)}) * R^{(N)} \tag{2.3}$$

The map $\mathcal{R}'^{(M+1)}$ associated to $R'^{(M+1)}$ acts on a generic M-comb $O^{(M)}$ as

$$\begin{aligned} \mathcal{R}'^{(M+1)}(O^{(M)}) &= \mathfrak{C}^{-1}(R'^{(M+1)})(O^{(M)}) = R'^{(M+1)} * O^{(M)} = \\ &= \mathfrak{C}(\mathcal{S}^{(N\to M+1)}) * R^{(N)} * O^{(M)} = \\ &= \mathfrak{C}(\mathcal{S}^{(N\to M+1)}) * (R^{(N)} \otimes O^{(M)}) \end{aligned} \tag{2.4}$$

Thus the map $\mathcal{S}^{(N\to M+1)}$ induces a map $\mathcal{S}^{(N\times M\to 1)}$ from tensor product operators into 1-combs defined as

$$\mathcal{S}^{(N\times M\to 1)}(R^{(N)} \otimes O^{(M)}) \doteq \mathfrak{C}(\mathcal{S}^{(N\to M+1)}) * (R^{(N)} \otimes O^{(M)}). \tag{2.5}$$

This map sends tensor product operators into a 1-comb, which is deterministic if $R^{(N)}$ and $O^{(N)}$ are deterministic. On the other hand, given a map $\mathcal{S}^{(N\times M\to 1)}$, we can define

$$\mathcal{S}^{(N\to M+1)}(R^{(N)}) \doteq \mathfrak{C}(\mathcal{S}^{(N\times M\to 1)}) * R^{(N)}. \quad (2.6)$$

Clearly, if $\mathcal{S}^{(N\times M\to 1)}$ sends tensor product of deterministic combs into deterministic 1-combs, then $\mathcal{S}^{(N\to M+1)}$ is deterministic. ■

The theorem can be easily generalized to $(N \to (M \to P))$-maps. Indeed, the essential part of the proof is the associativity of the link product of three Choi-Jamiołkowski operators, written for the special case in which the two operators $R^{(N)}$ and $O^{(M)}$ have no spaces in common[1]

$$\left(\mathfrak{C}\left(\mathcal{S}^{(N\to(M\to P))}\right) * R^{(N)}\right) * O^{(M)} = \mathfrak{C}\left(\mathcal{S}^{(N\to(M\to P))}\right) * \left(R^{(N)} \otimes O^{(M)}\right) \quad (2.7)$$

which immediately leads to

$$N \to (M \to P) \cong N \times M \to P. \quad (2.8)$$

An $N \times M$ pair has globally $N + M$ teeth, so that every admissible map on $N \times M$ pairs accepts as input an object with $N + M$ teeth. However, from an accurate reading of Theorem 2.1.1, it was realised in Ref. [31] that maps admissible on pairs are not maps of type $N \to 1$ for any N, since they are in correspondence with higher-order types. The hierarchy of higher-order quantum maps collapses on the comb level only if one states the hypothesis of *compatibility with remote connections* [29], accordingly to which one must fix the causal order of the teeth constituting the $N\times M$ pair before inputting the pair itself into an admissible map. Indeed, under this hypothesis, any uncurried $N \to M$ map must be a map of type $(N + M - 1) \to 1$ for some ordering of the $(N + M - 1)$ teeth belonging to the input comb. These maps can be performed resorting to Theorem 1.4.2 by linking the $(N+M-1)$ input comb with an appropriate $(N+M)$-comb. In conclusion, we can say that the hypothesis of compatibility with remote connections guarantees that the hierarchy of quantum maps collapses on the comb level. Nevertheless, if one introduces the hypothesis of compatibility with remote connections, then one prevents a genuine treatment of higher-order quantum maps.

What we have just said implies that we are not allowed to simulate those maps of type $N \to M$ that are not in correspondence to a map of type $P \to 1$ through an ordinary quantum circuit – namely a circuit satisfying the four prescriptions required by the Quantum Circuit Model of computation. We will provide in Sec. 2.3 a significant example of a map that cannot be achieved through an ordinary quantum circuit. This map is the switch of two quantum channels.

[1] In this case the link product operator coincides with the tensor product operator, as a consequence of Eq. (A.22).

2.2 The largest natural domain of uncurried $(1 \rightarrow 2)$-maps

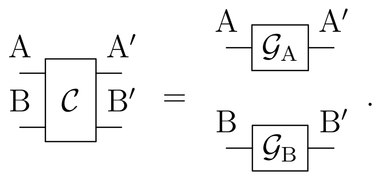
We now desire to individuate the largest natural domain on which uncurried $(1 \rightarrow 2)$-maps maps can be defined. Thanks to some results proved in Refs. [30, 31] we will see that uncurried $(1 \rightarrow 2)$-maps are well defined on deterministic and probabilistic quantum no-signalling channels. This set will also turn out to be the largest natural domain on which uncurried $(1 \rightarrow 2)$-maps are well defined.

Broadly speaking, what we have just claimed will be proved stating firstly the admissibility conditions of uncurried $(1 \rightarrow 2)$-maps on factorizable channels and then providing suitable theorems implying that such admissibility conditions guarantee the admissibility of uncurried $(1 \rightarrow 2)$-maps on no-signalling channels.

Before taking on this domain characterisation problem, we have to provide a formal definition of *factorizable, localizable* and *no-signalling* channels, along with some properties of them which are essential to achieve our goal.

2.2.1 Factorizable, separable, localizable and no-signalling channels

Definition 2.2.1. *A bipartite quantum channel* $\mathcal{C} : \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{B}) \rightarrow \mathcal{L}(\mathrm{A}') \otimes \mathcal{L}(\mathrm{B}')$ *is* factorizable *if it can be realised by local independent operations on* A *and* B, *namely if there exist two quantum channels* $\mathcal{G}_\mathrm{A} : \mathcal{L}(\mathrm{A}) \rightarrow \mathcal{L}(\mathrm{A}')$ *and* $\mathcal{G}_\mathrm{B} : \mathcal{L}(\mathrm{B}) \rightarrow \mathcal{L}(\mathrm{B}')$ *such that* $\mathcal{C} = \mathcal{G}_\mathrm{A} \otimes \mathcal{G}_\mathrm{B}$. *Pictorially:*



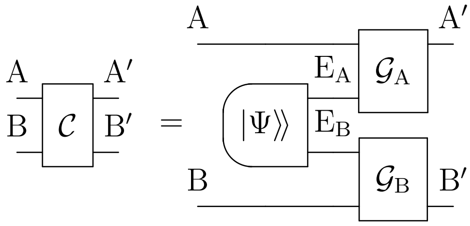
(2.9)

The Choi-Jamiołkowski operator of a bipartite factorizable channel as in Def. 2.2.1 is a factorizable operator in $\mathcal{L}(A' \otimes B' \otimes A \otimes B)$. This is easy proved exploiting the identity $|I\rangle\!\rangle\langle\!\langle I|_{A\otimes B, A'\otimes B'} = |I\rangle\!\rangle\langle\!\langle I|_{A,A'} \otimes |I\rangle\!\rangle\langle\!\langle I|_{B,B'}$ in the definition of the Choi-Jamiołkowski operator of the channel itself.

We also notice that the set of bipartite factorizable channels is not closed under linear combination. This will forbid us to choose it as a well defined domain for any linear map.

Definition 2.2.2. Separable *channels are convex combination of bipartite quantum operation, namely channels whose Choi-Jamiołkowski operator is separable with respect to* $\mathcal{L}(A' \otimes A) \otimes \mathcal{L}(B' \otimes B)$.

Definition 2.2.3. *A channel* $\mathcal{C} : \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{B}) \to \mathcal{L}(\mathrm{A}') \otimes \mathcal{L}(\mathrm{B}')$ *is* localizable *if it can be realised by local operations on* A *and* B *with a shared entangled ancilla on a couple of d-dimensional systems* $\mathrm{E_A}, \mathrm{E_B}$ *in a generic state* $|\Psi\rangle\rangle$ *but without communication:*



. (2.10)

Definition 2.2.4. *A bipartite quantum channel* $\mathcal{C} : \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{B}) \to \mathcal{L}(\mathrm{A}') \otimes \mathcal{L}(\mathrm{B}')$ *is* $\mathrm{A} \nrightarrow \mathrm{B}'$ no-signalling *if*

$$\mathrm{Tr}_{\mathrm{A}'}[R_\mathcal{C}] = I_\mathrm{A} \otimes S_{\mathrm{BB}'}. \tag{2.11}$$

where $S_{\mathrm{BB}'}$ *is the Choi-Jamiołkowski operator of some channel* $\mathcal{S} : \mathcal{L}(B) \to \mathcal{L}(B')$.

We say that $\mathcal{C}$ *is* no-signalling *if it is both* $\mathrm{A} \nrightarrow \mathrm{B}'$ *no-signalling and* $\mathrm{B} \nrightarrow \mathrm{A}'$ *no-signalling.*

Unlike the set of factorizable channels, the set of no-signalling channels is a closed set under linear combination. This property is enjoyed thanks to the linearity of the trace operation. In fact if $\mathcal{E} : \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{B}) \to \mathcal{L}(\mathrm{A}') \otimes \mathcal{L}(\mathrm{B}')$ is no-signalling and $\mathcal{E}' : \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{B}) \to \mathcal{L}(\mathrm{A}') \otimes \mathcal{L}(\mathrm{B}')$ is also no-signalling we have the following

$$\begin{array}{c} \mathrm{Tr}_{\mathrm{A}'}[R_\mathcal{E}] = I_\mathrm{A} \otimes S_{\mathrm{BB}'} \\ \mathrm{Tr}_{\mathrm{A}'}[R_{\mathcal{E}'}] = I_\mathrm{A} \otimes S'_{\mathrm{BB}'} \end{array} \Longrightarrow \begin{array}{rcl} \mathrm{Tr}_{\mathrm{A}'}[R_\mathcal{E} + R_{\mathcal{E}'}] & = & I_\mathrm{A} \otimes S_{\mathrm{BB}'} + I_\mathrm{A} \otimes S'_{\mathrm{BB}'} \\ & = & I_\mathrm{A} \otimes (S_{\mathrm{BB}'} + S'_{\mathrm{BB}'}) \,. \end{array} \tag{2.12}$$

A characterisation of no-signalling channels is given in Ref. [31, 33] by:

Theorem 2.2.1. *The following are equivalent:*

1. *The channel* $\mathcal{C} : \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{B}) \to \mathcal{L}(\mathrm{A}') \otimes \mathcal{L}(\mathrm{B}')$ *is no-signalling*

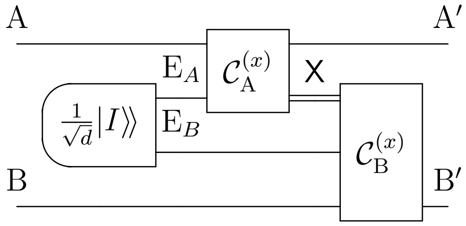
2. *There are equivalent d-dimensional quantum systems* $\mathrm{E_A}, \mathrm{E_B}$*, instruments* $\{\mathcal{C}^{(x)}_\mathrm{A}\}_{x \in \mathsf{X}}$ *and* $\{\mathcal{D}^{(x)}_\mathrm{B}\}_{x \in \mathsf{X}}$ *with outcome space* X*, and channels* $\mathcal{C}^{(x)}_\mathrm{B}$ *and* $\mathcal{D}^{(x)}_\mathrm{A}$ *for each* $x \in \mathsf{X}$ *with*

$$\begin{aligned} \mathcal{C}^{(x)}_\mathrm{A} &: \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{E_A}) \to \mathcal{L}(\mathrm{A}') \\ \mathcal{C}^{(x)}_\mathrm{B} &: \mathcal{L}(\mathrm{B}) \otimes \mathcal{L}(\mathrm{E_B}) \to \mathcal{L}(\mathrm{B}') \\ \mathcal{D}^{(x)}_\mathrm{B} &: \mathcal{L}(\mathrm{B}) \otimes \mathcal{L}(\mathrm{E_B}) \to \mathcal{L}(\mathrm{B}') \\ \mathcal{D}^{(x)}_\mathrm{A} &: \mathcal{L}(\mathrm{A}) \otimes \mathcal{L}(\mathrm{E_A}) \to \mathcal{L}(\mathrm{A}') \end{aligned} \tag{2.13}$$

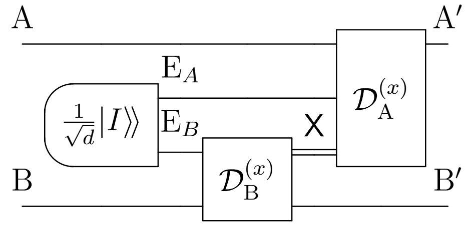
such that

$$\begin{aligned}\mathcal{C} &= \sum_{x \in X} \mathcal{C}_{\mathrm{B}}^{(x)} \circ \mathcal{C}_{\mathrm{A}}^{(x)} \left(d^{-1} |I\rangle\!\rangle\langle\!\langle I|_{\mathrm{E}_A,\mathrm{E}_B} \right) \\ &= \sum_{x \in \mathrm{X}} \mathcal{D}_{\mathrm{A}}^{(x)} \circ \mathcal{D}_{\mathrm{B}}^{(x)} \left(d^{-1} |I\rangle\!\rangle\langle\!\langle I|_{\mathrm{E}_A,\mathrm{E}_B} \right) ,\end{aligned} \tag{2.14}$$

namely, $\mathcal{C}$ has the two equivalent circuital realisations



, (2.15)



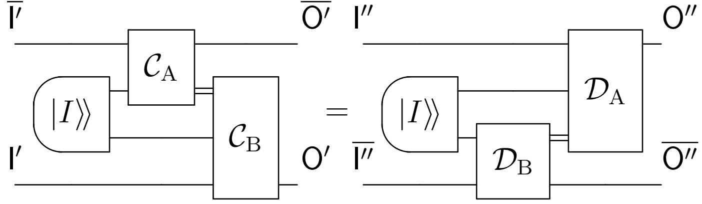
. (2.16)

The set of localizable channels is a proper subset of the set of no-signalling channels. This is proven through Theorem 2.2.1 itself, which shows that the most general no-signalling channel differs from a localizable channel because it also admits a single round of classical communication, with the constraint that it must be possible to implement the channel exploiting communication in either directions.

For a multipartite channel satisfying two different no-signalling conditions, an analog of Theorem 2.2.1 holds. In fact, let us consider a a channel $\mathcal{C}$ with input systems labelled by a set of indices I and output systems labelled by a set O. Suppose that $\mathcal{C}$ satisfies the following no-signalling conditions

$$\begin{aligned}\mathrm{Tr}_{\mathsf{O}'}[R_{\mathcal{C}}] &= I_{\mathsf{I}'} \otimes S_{\overline{\mathsf{O}'} \cup \overline{\mathsf{I}'}} \\ \mathrm{Tr}_{\mathsf{O}''}[R_{\mathcal{C}}] &= I_{\mathsf{I}''} \otimes T_{\overline{\mathsf{O}''} \cup \overline{\mathsf{I}''}}\end{aligned} \quad , \tag{2.17}$$

for certain subsets $\mathsf{I}', \mathsf{I}'' \subseteq \mathsf{I}$ and $\mathsf{O}', \mathsf{O}'' \subseteq \mathsf{O}$, where $\overline{\mathsf{S}}$ represents the set complement of S, and for suitable Choi-Jamiołkowski operators S and T. Following the proof of Theorem 2.2.1, we can show that two circuits realising $\mathcal{C}$ are



. (2.18)

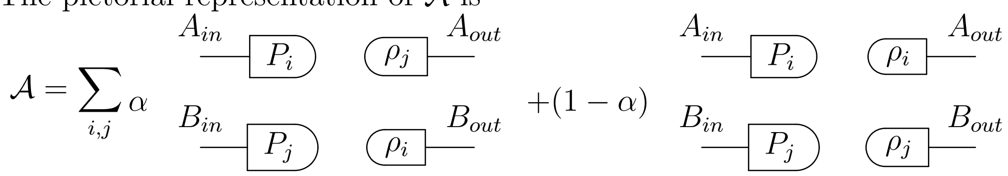
In general the subsets $\mathsf{I}', \mathsf{I}''$ are not a partition of I. In this case we have that the circuits cannot be realised partitioning the systems between the two local parties A and B. In particular the input systems in $\overline{\mathsf{I}'} \cap \overline{\mathsf{I}''}$ are always assigned to the party which sends the classical message, and input systems in $\mathsf{I}' \cap \mathsf{I}''$ are assigned to the party which receives the classical message (and similarly for output systems). One can also consider more complex scenarios, *i.e.* channels with more than two no-signalling conditions of the kind in Eq. (2.17), or channels with nested conditions, for example when the Choi-Jamiołkowski operators S and T in Eq. (2.17) satisfy no-signalling conditions on their own. However the analysis of the classical communication required in these cases is complicated, and it is an open problem.

We also report that there exists a simple crucial example of a separable no-signalling channel which is not a localizable operation, which has been reported in Ref. [36]. This is an implementation of the PR box introduced in Ref. [35].

Furthermore we can provide an example of a channel $\mathcal{A} : \mathcal{L}(A_{in} \otimes B_{in}) \to \mathcal{L}(A_{out} \otimes B_{out})$ that is both separable and signalling. In fact let $\{P_i\}$ be a complete family of Choi-Jamiołkowski operators with one dimensional output space, $\{\rho_i\}$ be a complete family of Choi-Jamiołkowski operators with one dimensional input space and $\alpha \in [0, 1]$. The Choi-Jamiołkowski operator of the claimed channel is given by

$$\mathrm{A} = \sum_{i,j} \alpha \left(P_i \otimes \rho_j\right) \otimes \left(P_j \otimes \rho_i\right) + (1-\alpha)\left(P_i \otimes \rho_i\right) \otimes \left(P_j \otimes \rho_j\right). \tag{2.19}$$

The pictorial representation of $\mathcal{A}$ is



(2.20)

It clearly follows from Eq. (2.19) that $\mathcal{A}$ is separable. Nevertheless A does not satisfy the property in Eq. (2.11) or an analogue thereof changing the roles of A_{in} with B_{in} and of A_{out} with B_{out}.

No-signalling and localizable channels also enjoy the remarkable property expressed in the following [31]

Theorem 2.2.2 (Semigroupoid property)**.** *Consider bipartite channels*

$$\begin{aligned} \mathcal{S} &\in \mathcal{L}(\mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_1), \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_3)), \\ \mathcal{T} &\in \mathcal{L}(\mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_3), \mathcal{L}(\mathcal{H}_4 \otimes \mathcal{H}_5)). \end{aligned} \tag{2.21}$$

If they are both no-signalling (localizable), then their composition $\mathcal{T} \circ \mathcal{S}$ is no-signalling (localizable).

We used the term *semigroupoid* instead of semigroup because a in a semigroup we require the every pair of elements are composable. Maps, on the other hand, can be composed only if input and output spaces match.

2.2.2 Admissibility constraints for uncurried $(1 \to 2)$-maps

We now come to individuate the constraints that deterministic uncurried $(1 \to 2)$-maps must satisfy to be compatible with quantum mechanics. Theorem 2.1.1 states that deterministic uncurried $(1 \to 2)$-maps are nothing but deterministic (1×1)-maps, namely maps that send two bipartite factorizable channels in one deterministic channel. The set of bipartite factorizable channels is not a closed set under linear combination, the latter being a fundamental property that a valid domain of every higher-order quantum maps is supposed to satisfy, as it will be soon pointed out.

The smallest set which contains all the bipartite factorizable channels and which is contemporarily closed under linear combination is clearly the span of the same channels. We now discuss the admissibility conditions for the elements of that span.

A deterministic map $\mathcal{S}$ transforming bipartite factorizable channels into channels is required to be linear on the span of bipartite factorizable channels, in order to be compatible with the probabilistic structure of quantum mechanics. This follows from these two properties:

- If we apply the map to the channel $\mathcal{C} = \sum_i p_i \mathcal{A}_i \otimes \mathcal{B}_i$ – corresponding to a random choice of bipartite factorizable channels – the output must be a random choice of bipartite factorizable channels with the same probabilities, *i.e.* $\mathcal{S}(\mathcal{C}) = \sum_i p_i \mathcal{S}(\mathcal{A}_i \otimes \mathcal{B}_i)$;
- It must hold $\mathcal{S}(p\mathcal{C}) = p\,\mathcal{S}(\mathcal{C})$.

Another admissibility requisite for a deterministic (1×1)-map is to be locally completely positive, whose meaning is set forth by the following general definition[2]:

Definition 2.2.5. *A map $\mathcal{S}^{N \times M \to 1} : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$ is locally CP if $\mathcal{S} \otimes \mathcal{I}$ is positive on positive tensor product operators $R_1 \otimes R_2$ with $R_1 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ and $R_2 \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{K}_2)$.*

The property in Def. 2.2.5 is required if we want $\mathcal{S}$ to produce a legitimate output channel when applied locally on an extended input, as it was discussed in Sec. 1.4 and in Ref. [27, 31]. A bipartite factorizable map that enjoys this property is trivially CP too.

[2]In Def. 2.2.5 the Hilbert spaces on which the N- and M-combs are defined are to be intended to possess their natural tensor product structure, namely $\mathcal{H} \doteq \otimes_{i=0}^{N-1} \tilde{\mathcal{H}}_i$ and $\mathcal{H} \doteq \otimes_{i=0}^{M-1} \tilde{\mathcal{H}}_i$.

The third and last admissibility property for a deterministic (1×1)-map is normalisation preserving, namely if $\mathcal{S}^{1\times1\to1} : \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3) \otimes \mathcal{L}(\mathcal{H}_4, \mathcal{H}_5) \to \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ is deterministic, then it must hold

$$I_A * S * (A \otimes B) = I_B, \tag{2.22}$$

where S is the Choi-Jamiołkowski operator of $\mathcal{S}$ [29]. We now prove that if a deterministic map $\mathcal{S}^{1\times1\to1}$ is normalisation preserving then it preserves normalisation also when applied locally on an extended input.

Lemma 2.2.1. *Consider a deterministic map* $\mathcal{S}^{1\times1\to1} : \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3) \otimes \mathcal{L}(\mathcal{H}_4, \mathcal{H}_5) \to \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$*. Then* $\mathcal{S} \otimes \mathcal{I} : \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_2, \mathcal{H}_1 \otimes \mathcal{H}_3) \otimes \mathcal{L}(\mathcal{H}_4 \otimes \mathcal{H}_6, \mathcal{H}_5 \otimes \mathcal{H}_7) \to \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_A \otimes \mathcal{H}_6, \mathcal{H}_1 \otimes \mathcal{H}_B \otimes \mathcal{H}_7)$ *is normalisation preserving if* $\mathcal{S}$ *is normalisation preserving.*

Proof. Let us define $S \doteq \mathfrak{C}(\mathcal{S})$, $R_1 \doteq \mathfrak{C}(\mathcal{R}_1)$, $R_2 \doteq \mathfrak{C}(\mathcal{R}_2)$, $C_{0A6,1B7} \doteq S * R_1 * R_2$, where $\mathcal{R}_1 : \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_3)$ and $\mathcal{R}_2 : \mathcal{L}(\mathcal{H}_4 \otimes \mathcal{H}_6) \to \mathcal{L}(\mathcal{H}_5 \otimes \mathcal{H}_7)$ are two quantum channels. After having traced on the output spaces $\mathcal{H}_1$, $\mathcal{H}_B$, $\mathcal{H}_7$, the Choi-Jamiołkowski operator of the network is

$$\begin{aligned} X &\doteq C * I_1 * I_B * I_7 \\ &= (S * I_B) * (R_1 * I_1) \otimes (R_2 * I_7). \end{aligned} \tag{2.23}$$

Since S, R_1, R_2 are deterministic Choi-Jamiołkowski operators, Eq. (2.23) becomes

$$\begin{aligned} X &= I_A * I_0 * I_6 \\ &= I_{A06}. \end{aligned} \tag{2.24}$$

This completes the proof. ■

We synthesise the discussion we formulated up to now in this section stating the following

Admissibility conditions for deterministic (1×1)-maps

A deterministic (1×1)-map is admissible if it is

1. Linear;
2. CP;
3. Normalisation preserving.

We now move on to the admissibility conditions for probabilistic (1×1)-maps. These conditions are not difficult to find, because they coincide with the ones required by a deterministic (1×1)-map to be admissible, except obviously normalisation preserving. This is straightforward to understand remembering that since the higher-order map $\mathcal{I} = \frac{\mathcal{I}}{r_{\mathcal{I}_1}} \otimes \frac{\mathcal{I}}{r_{\mathcal{I}_2}}$ is factorized and deterministic, then every Choi-Jamiołkowski operator X of a bipartite higher-order map $\mathcal{X}$ can be re-scaled to a probabilistic one through a proper coefficient $\lambda \leqslant \frac{1}{(d_{\mathcal{I}_1} d_{\mathcal{I}_2})\alpha_r(X)}$, where $\alpha_r(X)$ is the greater eigenvalue of X. Thus we can enunciate the following:

Admissibility conditions for probabilistic (1×1)-maps

A probabilistic (1×1)-map is admissible if it is

1. Linear;

2. CP.

2.2.3 Individuation of the largest natural domain of uncurried $(1 \to 2)$-maps

We begin this section observing that the set of factorizable channels is a subset of localizable channels. Hence, they are obviously no-signalling. This proves that every map which is admissible on no-signalling channels is automatically admissible on pairs of factorizable channels. This is true both for probabilistic and for deterministic types of maps.

Moreover it holds the converse too, namely that every map which is admissible on factorizable channels is also admissible on no-signalling channels. One can become aware of this exploiting Theorem 14 of Ref. [30], in which it is proven that if $\Lambda : \mathcal{L}(\otimes_{i=1}^m \mathrm{X}_i) \to \mathcal{L}(\otimes_{i=1}^m \mathrm{A}_i)$ is a no-signalling operation, then its Choi-Jamiołkowski operator $J(\Lambda)$ belongs to $\otimes_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{Q}_\mathsf{i}$, where each set $\mathsf{Q}_\mathsf{i} \subset \mathsf{H}(\mathrm{A}_i \otimes \mathrm{X}_i)$ denotes the subspace of Hermitian operators $J(\Phi)$ for which $\Phi : \mathcal{L}(\mathrm{X}_i) \to \mathcal{L}(\mathrm{A}_i)$ is a trace-preserving super-operator, or a scalar multiple thereof. We now present a slightly different form of that theorem, that can be found in [31].

Theorem 2.2.3. *Consider a Hermitian preserving map*

$$\mathcal{X} \in \mathcal{L}(\mathcal{L}(\mathrm{A} \otimes \mathrm{B} \otimes \mathrm{C}), \mathcal{L}(\mathrm{A}' \otimes \mathrm{B}' \otimes \mathrm{C}')) \tag{2.25}$$

along with its Choi-Jamiołkowski operator $X = \mathfrak{C}(\mathcal{X})$. Then the following are equivalent

1.
$$X \in \mathrm{comb}(\mathrm{A}, \mathrm{A}') \otimes \mathrm{comb}(\mathrm{B}, \mathrm{B}', \mathrm{C}, \mathrm{C}') \cap \mathrm{comb}(\mathrm{ABC}, \mathrm{A}'\mathrm{B}'\mathrm{C}') \tag{2.26}$$

2.

$$X \in \mathrm{comb}(\mathrm{A}, \mathrm{A}', \mathrm{B}, \mathrm{B}', \mathrm{C}, \mathrm{C}') \cap \mathrm{comb}(\mathrm{B}, \mathrm{B}', \mathrm{A}, \mathrm{A}', \mathrm{C}, \mathrm{C}') \cap \\ \cap \mathrm{comb}(\mathrm{B}, \mathrm{B}', \mathrm{C}, \mathrm{C}', \mathrm{A}, \mathrm{A}'). \tag{2.27}$$

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1)

Let $\{E^i\}_{i\in\mathsf{J}}$ be an operator basis for $\mathcal{L}(\mathrm{A} \otimes \mathrm{A}')$. Then one can find operators X^i such that

$$X = \sum_{i\in\mathsf{J}} E^i_{\mathrm{AA}'} \otimes X^i_{\mathrm{BB}'\mathrm{CC}'} \tag{2.28}$$

Let us consider a dual set of operators $\tilde{E}^j$, such that

$$\tilde{E}^j * E^i = \delta_{ij} k_j \tag{2.29}$$

for some real numbers k_j. Exploiting the normalisation of X we have that

$$(X * I_{\mathrm{C}'}) * \tilde{E}^j_{\mathrm{AA}'} = (I_C \otimes Y_{\mathrm{AA}'\mathrm{BB}'}) * \tilde{E}^j_{\mathrm{AA}'} = I_{\mathrm{C}} \otimes (Y_{\mathrm{AA}'\mathrm{BB}'} * \tilde{E}^j_{\mathrm{AA}'}), \tag{2.30}$$

for some operator Y. On the other hand, by the properties of link product we can also write

$$(X * \tilde{E}^j_{\mathrm{AA}'}) * I_{\mathrm{C}'} = \left[\left(\sum_{i\in\mathsf{J}} E^i_{\mathrm{AA}'} \otimes X^i_{\mathrm{BB}'\mathrm{CC}'} \right) * \tilde{E}^j_{\mathrm{AA}'} \right] * I_{\mathrm{C}'} = \\ = \sum_{i\in\mathsf{J}} E^i_{\mathrm{AA}'} * \tilde{E}^j_{\mathrm{AA}'} \otimes X^i_{\mathrm{BB}'\mathrm{CC}'} * I_{\mathrm{C}'} = k_j X^j_{\mathrm{BB}'\mathrm{CC}'} * I_{\mathrm{C}'}. \tag{2.31}$$

Posing $R^j_{\mathrm{BB}'} \doteq k_j^{-1}(Y_{\mathrm{AA}'\mathrm{BB}'} * \tilde{E}^j_{\mathrm{AA}'})$ we can conclude that

$$X^j_{\mathrm{BB}'\mathrm{CC}'} * I_{\mathrm{C}'} = I_{\mathrm{C}} \otimes R^j_{\mathrm{BB}'}. \tag{2.32}$$

with $R^j_{\mathrm{BB}'} * I_{\mathrm{B}'}$ proportional to I_{B}. Thus we have proved that each X^j is proportional to an element in $\mathrm{comb}(\mathrm{B}, \mathrm{B}', \mathrm{C}, \mathrm{C}')$. Now, choosing $\mathsf{J}' \subset \mathsf{J}$ such that $\{X^j\}_{j\in\mathsf{J}'}$ is a maximally linearly independent subset, we can write

$$X = \sum_{j\in\mathsf{J}'} Z^j_{\mathrm{AA}} \otimes X^j_{\mathrm{BB}'\mathrm{CC}'} \tag{2.33}$$

for suitable operators Z^j. The same argument proves that each Z^j is proportional to an element in $\mathrm{comb}(\mathrm{A}, \mathrm{A}')$. ■

Thanks to what we have shown up to now in this section, we are in position to assert the following

Lemma 2.2.2. *The span of factorizable channels coincides with the span of no-signalling channels.*

We now state the following

Theorem 2.2.4. *Let $\mathcal{S}$ be an admissible map on the span of bipartite factorizable channels. Then $\mathcal{S}$ is also admissible on the span of no-signalling channels.*

The proof of Theorem 2.2.4 becomes straightforward if we show that a $\mathcal{S}^{N\times M\to 1}$ map being locally CP on no-signalling channels is also a CP map on the same, as stated in the following

Lemma 2.2.3. *Consider a map $\mathcal{S}^{N\times M\to 1} : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$ as in Def. 2.2.5. Then $\mathcal{S}^{N\times M\to 1}$ is also a CP map.*

Proof. The Choi-Jamiołkowski operator of $\mathcal{S}^{N\times M\to 1}$ is

$$S \doteq \mathcal{S}^{N\times M\to 1} \otimes \mathcal{I}\left(|I\rangle\!\rangle\langle\!\langle I|_{\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{K}_1\otimes\mathcal{K}_2}\right). \tag{2.34}$$

Since $|I\rangle\!\rangle\langle\!\langle I|_{\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{K}_1\otimes\mathcal{K}_2} = |I\rangle\!\rangle\langle\!\langle I|_{\mathcal{H}_1,\mathcal{K}_1} \otimes |I\rangle\!\rangle\langle\!\langle I|_{\mathcal{H}_2,\mathcal{K}_2}$, the locally CP property of $\mathcal{S}^{N\times M\to 1}$ implies that the Choi-Jamiołkowski operator S is positive, hence the map is also CP. ■

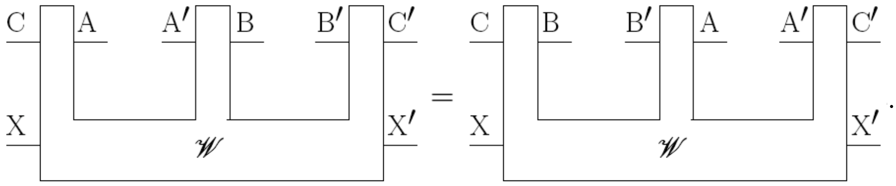
Proof of Theorem 2.2.4. From Lemma 2.2.2 it comes that linearity of $\mathcal{S}$ on one of that spans implies linearity of $\mathcal{S}$ on the other. Instead, combining Lemma 2.2.2 with the linearity of the trace one has that if $\mathcal{S}$ is normalisation preserving, then a normalised map on one of that spans is also normalised on the other. If $\mathcal{S}$ is locally CP on one of that spans, then $\mathcal{S}$ is also locally CP on the other span. We noticed in Sec. 2.2.2 that a map being locally CP on the span of bipartite factorizable channels is also a CP map. Lemma 2.2.3 states that the same holds for a map defined on the span of no-signalling channels. Thus a map which is linear, CP and normalisation preserving on the span of factorizable channels enjoys the same properties on the span of no-signalling channels. This proves the thesis of the theorem. ■

2.3 The No-Switch theorem

We pointed out in Sec. 2.1 that maps admissible on pairs are not maps of type $N \to 1$ for any N, unless one postulates the hypothesis of compatibility with remote connections. We now provide a notable example of map which is admissible on no-signalling combs and yet cannot be traced back to a $N \to 1$ map. This is the switch map $\mathcal{W}$, which takes as input a no-signalling comb and outputs a bipartite channel

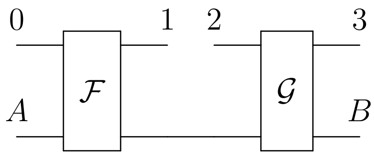
$$\mathcal{W} : \mathrm{comb}(\mathrm{A}, \mathrm{A}', \mathrm{B}, \mathrm{B}') \cap \mathrm{comb}(\mathrm{B}, \mathrm{B}', \mathrm{A}, \mathrm{A}') \longrightarrow \mathrm{comb}(\mathrm{XC}, \mathrm{X}'\mathrm{C}'), \tag{2.35}$$

where X and X′ are qubit systems. Pictorially we have



(2.36)

In the simplest version of the switch, $\mathrm{A}, \mathrm{A}', \mathrm{B}, \mathrm{B}', \mathrm{C}, \mathrm{C}'$ are quantum systems with the same dimension. The map $\mathcal{W}$ is defined as follows: on a pair of combs (F, G) representing channels $(\mathcal{F}, \mathcal{G})$ (*i.e.* on an object of pair type), it gives the composition $\mathfrak{C}(\mathcal{F} \circ \mathcal{G})$ or $\mathfrak{C}(\mathcal{G} \circ \mathcal{F})$, depending whether the control qubit is $|0\rangle$ or $|1\rangle$ respectively. The map is then extended by linearity to every state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ on the control qubit. If the switch map is fed with the following input



(2.37)

then the Choi-Jamiołkowski operator of the switch map is

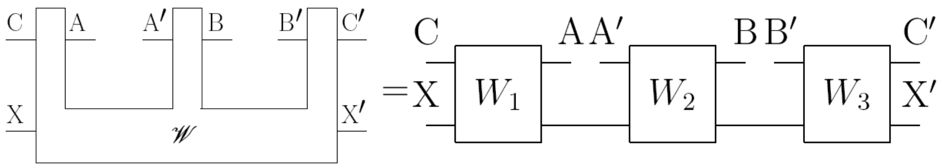
$$\begin{aligned} \mathcal{W}(F \otimes G) \;=\; & |I\rangle\rangle\langle\langle I|_{12} \, * \\ & * \left(|00\rangle_{AB} \otimes I_{01,23} + |11\rangle_{AB} \otimes E_{01,23}\right) \left(F_{01} \otimes G_{23}\right) \\ & \left(\langle 00|_{AB} \otimes I_{01,23} + \langle 11|_{AB} \otimes E_{01,23}\right), \end{aligned} \tag{2.38}$$

where $E_{01,23}$ is the Choi-Jamiołkowski operator that describes the mutual exchange of the two teeth F and G.

This map is clearly admissible on pairs, according to the admissibility conditions that we stated in Sec. 2.2.2. Thus the switch map is well-defined also on the set of no-signalling channels too, as it was proved in Sec. 2.2.2. Nevertheless, the switch cannot be implemented as quantum comb, as it is proved in the following [6]

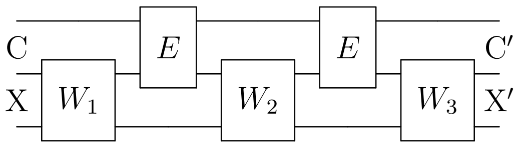
Theorem 2.3.1 (No-switch of boxes)**.** *The map $\mathcal{W}$ is not realisable as deterministic quantum comb.*

Proof. Suppose by absurd that $\mathcal{W}$ is realisable as deterministic quantum comb, *i.e.* it admits a circuital realisation as follows



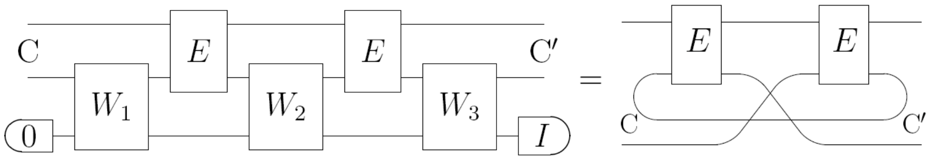
(2.39)

Then, we can apply it to a couple of linked swap gates as follows:



(2.40)

and obtain a properly normalised quantum channel. But, for $|\psi\rangle = |0\rangle$, the definition of the switch map leads to



(2.41)

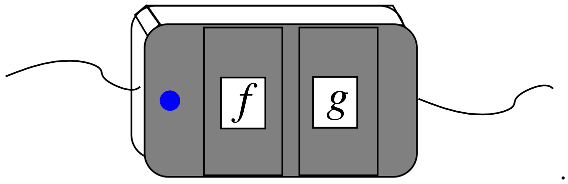
(where the control qubit X′ has been traced away). The right hand side of this equation does not satisfy the normalisation conditions for an admissible quantum comb. This contradiction implies that no such realisation as Eq. (2.39) exists. ■

Theorem 2.3.1 shows that the set of admissible maps on the span of no-signalling channels is strictly larger then the set of quantum combs. It is not known how to characterise this set of maps in terms of a universal set. It is proposed that the following holds:

Conjecture 1. *The set of admissible transformations on bipartite no-signalling channels is generated by the switch map* $\mathcal{W}$ *and the quantum combs.*

More generally, we can ask whether there is a universal set which generates every possible higher-order map, including all maps which are admissible on various sets of no-signalling combs. If Conjecture 1 is true, it is established a broader computational model than Deutsch's ordinary Quantum Circuit Model, whose set of universal gates is given *e.g.* by the set single qubit transformations, the `CNOT` gate and the `QSO`, namely the gate which implements the switch function.

Apart from the validity of Conjecture 1, we will refer to a quantum network in which any `QSO` is contained with the name *dynamical computational network,* to distinguish it from an ordinary quantum circuit. This definition was chosen since a simplified version of the `QSO` could be intuitively implemented by a machine with two slots [6], in which the user can plug two variable boxes $\boxed{f}$ and $\boxed{g}$ at his choice, as in the following figure:



The machine is programmed with the following code:

`Program 1 (SWITCH)`

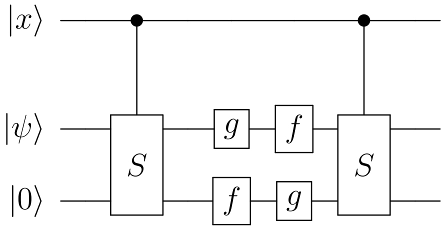
$$\texttt{if}\ |x\rangle = |1\rangle \quad \texttt{do}-\boxed{f}-\boxed{g}-$$
$$\texttt{if}\ |x\rangle = |0\rangle \quad \texttt{do}-\boxed{g}-\boxed{f}-.$$

We can imagine that the machine has movable wires inside, that can connect the boxes $\boxed{f}$ and $\boxed{g}$ in two possible ways depending on the state of the qubit $|x\rangle$, thus implementing the `SWITCH` function. Ordinary quantum circuits, however, do not have such movable wires. They can have controlled swap operations, but once a time-ordering between $\boxed{f}$ and $\boxed{g}$ has been chosen in the circuit, there is no way to reverse it.

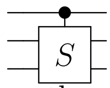
2.4 Switching channels through post-selected teleportation

Let us now focus our attention on the computational rules that characterise the Quantum Circuit Model – which were listed in Sec. 1.1 – and let us try to understand which are the constraints that prevent the feasibility of the `QSO` through an ordinary quantum circuit.

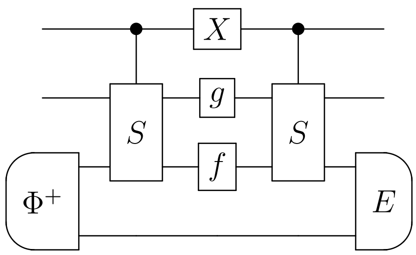
The first limitation arises from the fact that the two channels inputting the `QSO` are restricted to be called once, so that the circuit must contain boxes $\boxed{f}$ and $\boxed{g}$ only once (due to Rule 4) and in a definite time order (due to Rule 3). Indeed, a computational circuit that produces the same output of Program 1 actually exists, but it requires two calls to both oracles $\boxed{f}$ and $\boxed{g}$ as follows:



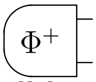
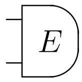
(2.42)

The circuit in Eq. (2.42) achieves the desired transformation over the qubit in the state $|\psi\rangle$ depending on the state of the control qubit $|x\rangle$. Here  is a control-swap gate, which exchanges the two input qubits depending on the state of the control qubit. Nevertheless, if the input are two black boxes $\boxed{f}$, $\boxed{g}$, the possibility of achieving two uses from a single one is ruled out by the no-cloning theorem for boxes [37]. Again, the limitation due to the single call constraint is strictly related to the "physical" nature of the unknown black boxes $\boxed{f}$ and $\boxed{g}$. If we knew what $\boxed{f}$ and $\boxed{g}$ are, we would be able to duplicate them, thus making possible the computation of the function $S\left(|x\rangle, \boxed{f}, \boxed{g}\right)$ through the circuit in Eq. (2.42).

Another factor that forbids the implementation of Program 1 as a computational circuit is the requirement that the program succeeds deterministically. Indeed, the prescriptions embedded in the Quantum Circuit Model do not hinder to achieve the same output of the `QSO` probabilistically. In particular, a computational circuit that uses post-selected teleportation[3] succeeds in the task with probability 1/4 is the following



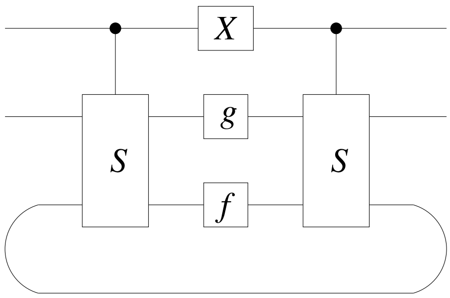
(2.43)

Here  is a maximally entangled state of two qubits – *i.e.* an element of the Bell basis – and  denotes the projection on $|\Phi^+\rangle\rangle$, which is an outcome of a Bell measurement. The gate $\boxed{X}$ represents the Pauli σ_X operation on the

[3] See *e.g.* Refs. [38, 39, 40] for particular details on this protocol.

control qubit. When the outcome E occurs in this circuit, we may say that the third qubit from the top has been teleported from the future back to the past. In this case it is easy to see that if the control qubit is in state $|1\rangle$ one obtains the sequence "$\boxed{f}$ followed by $\boxed{g}$" acting on the second input qubit, while if the control qubit is in state $|0\rangle$ one obtains the other sequence "$\boxed{g}$ followed by $\boxed{f}$". What is more, if one puts the control qubit in the superposition $\frac{1}{\sqrt{2}} |0\rangle + |1\rangle$, then one would get the superposition of the two orderings of the boxes, namely the output of the circuit is proportional to $U_f U_g |\psi\rangle \otimes |1\rangle + U_g U_f |\psi\rangle \otimes |0\rangle$, where $|\psi\rangle$ is the input state of the qubit in the middle wire, and U_f and U_g denote the unitary operators corresponding to boxes $\boxed{f}$ and $\boxed{g}$ respectively. Note, however, that the probability of achieving the controlled switch of $\boxed{f}$ and $\boxed{g}$ transforming N qubits goes to zero exponentially as 4^{-N} versus the number N of input qubits for each box. In fact, for each to be teleported qubit, Bob will hold the state that Alice wants to teleport only if Alice makes a projective measurement on the same entangled state she shares with Bob itself, that happens with probability $1/4$.

Revising the post-selected teleportation scheme, we devise that the switch of two boxes with an ordinary quantum circuit may be deterministically accomplished introducing a loop in the circuit, as it is showed in Eq. (2.44).



(2.44)

Nevertheless the loop represents a qubit that travels backward in time, thus violating causality as expressed by Rule 3.

In a certain sense, this simple example is complementary to the results of Ref. [41], whose authors showed that closed time-like curves do not improve tasks of first-order computation, like state discrimination. Here we have instead an impossible higher-order computation that would become reliable by a quantum circuit if a closed time-like curve were available. Note however, that the teleportation-based model of time travel considered here is different from the nonlinear model by Deutsch [42], which provided the framework for the results of Ref. [41].

Chapter 3

Programming dispositions and permutations of channels

We evaluate in this chapter the possibility of programming controlled dispositions and permutations of N unitary channels and their superposition through both an ordinary quantum circuit and a dynamical computational network.

We will start showing that the task of preparing all the possible disposition of N arbitrary unitary channels $\{U_i \,|\, i = 0, \ldots, (N-1)\}$ and superpositions of them acting on a qubit system is efficiently achievable implementing an ordinary quantum circuit made of two registers (the ancillary one that tallies $N \log N$ qubits and the fundamental one that tallies N qubits), in which $\mathbf{O}(N^2)$ elementary operations are performed and each channel is utilised N times.

We will then establish graphical rules to represent a dynamical computational network and we will build up a network which efficiently programs controlled permutations of N arbitrary unitary channels and superpositions of them. To accomplish this task, it will be proved that one should have at disposal an ancillary register made of $\frac{1}{2}N\,(N-1)$ qubits and as many `QSO`s. Unlike the previous case, here each input channel will be utilised once.

3.1 Programming dispositions of channels through ordinary quantum circuits

It is given the following task:

Task 1. *Let* $\mathcal{U} \doteq \{U_i \,|\, i = 0, \ldots, N-1\}$ *be a set of* N *unitary channels defined on the space* $\mathcal{H}$ *of a qubit system. Build an efficient quantum circuit that lets the state* $|\psi\rangle$ *of a generic qubit undergo one of the* N^N *possible dispositions of the* N *above introduced channels, depending on the state of a control register.*

We will show in Section 3.1.2 that Task 1 has a straightforward solution if one can solve the following

Task 2. *Let* $\mathcal{U} \doteq \{U_i \,|\, i = 0, \ldots, N-1\}$ *be a set of* N *unitary channels defined on the space* $\mathcal{H}$ *of a qubit system and let* $|\psi\rangle$ *be a generic state of a qubit. Build an efficient quantum circuit whose output is one of the* N *states* $U_i |\psi\rangle$*, once again according to the state of a control register.*

We will solve Task 2 in the following section.

3.1.1 The $\mathrm{C}^{(n)}(S)$ gate

We now outline the circuit that realises Task 2 in an efficient way, observing the prescriptions that characterise an ordinary quantum circuit, which were listed in Sec. 1.1).

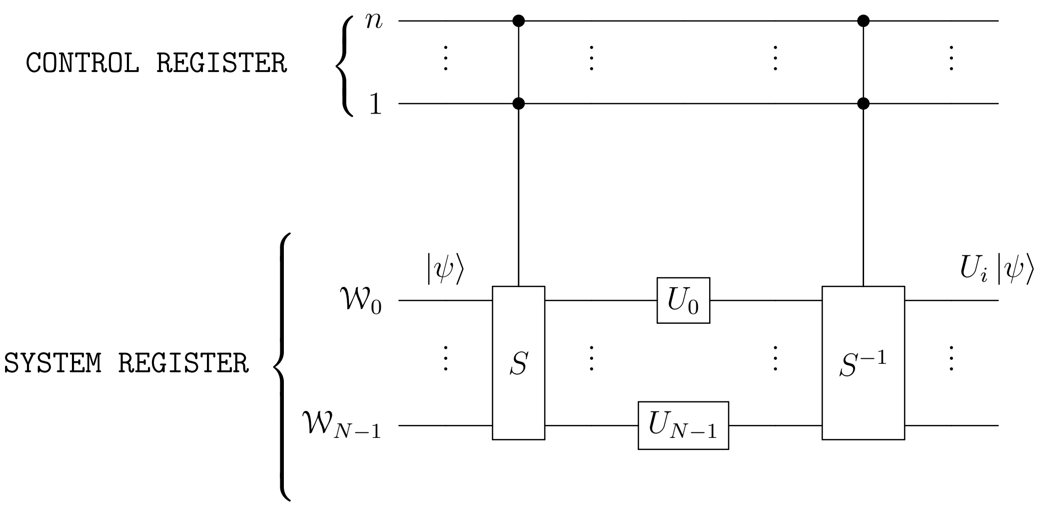
Let us define a bijective function $\mathcal{S} : \{U_i\} \to \{0,1\}^n$, with $n \doteq \log N$, which assigns to each unitary channel U_i the binary representation of its labelling number i, preceded by as many 0s as required to fill the string to n digits.

Now consider a circuit composed by two registers:

1. In the first one, made of n control qubits, the programmer inputs the desired control state, whose representation in the standard basis of $\mathcal{H}^{\otimes n}$ is given by $\mathcal{S}(U_i)$;

2. The second one is made of N qubits, initially prepared in the state $|\psi\rangle \otimes |0\rangle^{N-1} \in \mathcal{H}^{\otimes N}$, where $|\psi\rangle$ is the input state and $|0\rangle$ is a reference state of $\mathcal{H}$. In the following we will denote the jth wire of this register by $\mathcal{W}_j$ ($j = 0, \ldots, N-1$) or, alternatively, by $\mathcal{W}_{\mathcal{S}(U_j)}$.

Our task can be achieved if one could efficiently implement an *n-controlled swap gate* (from now on: $\mathrm{C}^{(n)}(S)$ gate). Indeed, it would be sufficient to send the system state from the wire $\mathcal{W}_0$ to the wire $\mathcal{W}_i$ entering the desired unitary channel U_i and then to swap back the wires $\mathcal{W}_i$ and $\mathcal{W}_0$ using the $\mathrm{C}^{(n)}(S^{-1})$ gate once the state $|\psi\rangle$ has undergone the channel U_i, eventually obtaining the wire $\mathcal{W}_0$ in the state $U_i |\psi\rangle$. This protocol is sketched in Eq. (3.1).



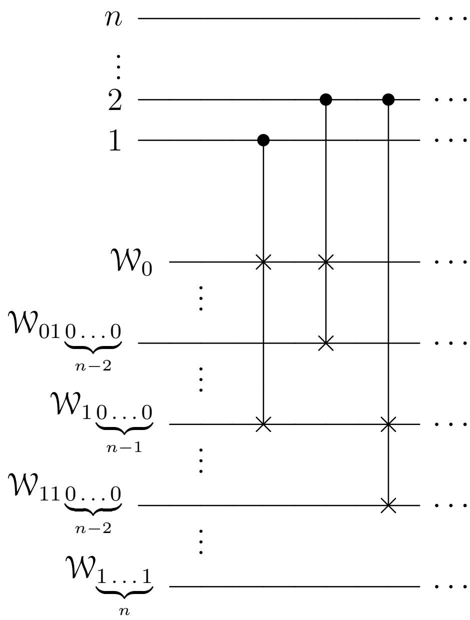
(3.1)

Such a gate can be realised resorting to $(N-1)$ single-controlled swap gates (also known as Fredkin gates), carrying out this program:

`Program 2 (C`$^{(\mathtt{n})}$`-SWAP)`

- Put a Fredkin gate controlled by the first control qubit, which swaps the wire $\mathcal{W}_{\underbrace{0\ldots0}_{n}}$ with the wire $\mathcal{W}_{1\underbrace{0\ldots0}_{n-1}}$;
- `for (k=1, k=n-1, k++)`
  Insert 2^k Fredkin gates controlled by the $(k+1)$th control wire that swap each system register's wire $\mathcal{W}_{\underbrace{s}_{k}\underbrace{0\ldots0}_{n-k-1}}$ with the wire $\mathcal{W}_{\underbrace{s}_{k}1\underbrace{0\ldots0}_{n-k-1}}$ for every k-bit string s.

The circuital representation emerging from the first steps of Program 2 is depicted in Eq. (3.2) for a generic value of n.



(3.2)

Let us count the number of Fredkin gates required to implement Program 2:

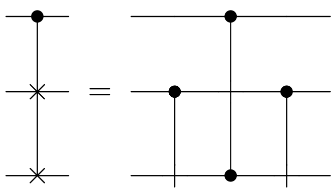
- the first step employs only one Fredkin gate;
- the kth step of the `FOR` cycle employs 2^k Fredkin gates.

So the whole program needs

$$\begin{aligned} \sum_{i=0}^{n-1} 2^k &= 2^n - 1 \\ &= N - 1 \end{aligned}$$

Fredkin gates to perform the $\mathrm{C}^{(n)}(S)$ gate.

Knowing that the following decomposition holds:



(3.3)

and that the optimal implementation of the Toffoli gate requires 6 `CNOT`s and 9 single qubit operations [43] we conclude that one needs $17(N-1)$ elementary operations[1] to set up a quantum circuit that performs the $\mathrm{C}^{(n)}(S)$ gate.

[1]We remember that the set of single qubit operations in conjunction with the `CNOT` gate constitutes an universal set of elementary operations [8].

Having explained how one can implement a $\mathrm{C}^{(n)}(S)$ gate in an efficient way using $\mathbf{O}(N)$ elementary operations, we can exploit this resource to build the circuit shown in Eq (3.2), which accomplishes Task 2 resorting to $34(N-1)$ elementary operations, viz. `CNOT` gates and single qubit operations.

Case $N = 8$, $n = 3$

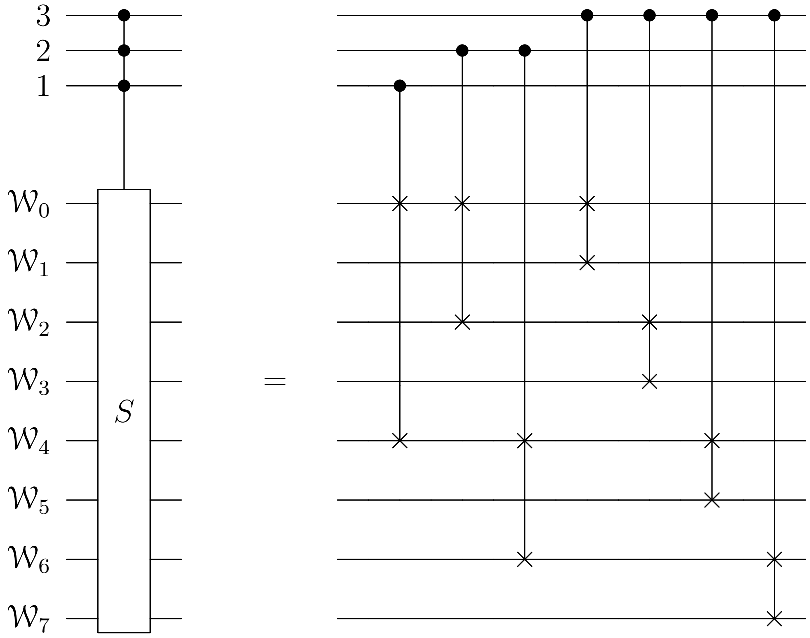
We now illustrate as an example how Task 2 is achieved when $N = 8$, $n = 3$.

Program 2 states that the $\mathrm{C}^{(3)}(S)$ gate is obtained placing 7 Fredkin gates, ordering them as they appear reading the second column of Table 3.1:

Table 3.1

Step of `PROGRAM "C`$^{(n)}$`-SWAP"`		Fredkin gates to be placed in the circuit	Control wire
1^{st}		$\mathrm{S}_{\mathcal{W}_0,\mathcal{W}_4}$	1^{st}
`FOR cycle`	`k = 1`	$\mathrm{S}_{\mathcal{W}_0,\mathcal{W}_2}$, $\mathrm{S}_{\mathcal{W}_4,\mathcal{W}_6}$	2^{nd}
	`k = 2`	$\mathrm{S}_{\mathcal{W}_0,\mathcal{W}_1}$, $\mathrm{S}_{\mathcal{W}_2,\mathcal{W}_3}$, $\mathrm{S}_{\mathcal{W}_4,\mathcal{W}_5}$, $\mathrm{S}_{\mathcal{W}_6,\mathcal{W}_7}$	3^{rd}

The circuit that realises the $\mathrm{C}^{(3)}(S)$ gate when $N = 8$ is eventually depicted in Eq. (3.4):



(3.4)

Writing down for every $i = 0, \ldots, 7$ a Gray code that connects the string $\mathcal{S}(U_0)$ to the string $\mathcal{S}(U_i)$ always starting to flip the appropriate digits from left to right, one reconstructs for every $i = 0, \ldots, 7$ the path that the system state follows before entering each box U_i. Table 3.2 synthesises all the possible cases.

Table 3.2

U_i	$\mathcal{S}(U_i)$	Gray code linking $\mathcal{S}(U_0)$ to $\mathcal{S}(U_i)$	Fredkin gates acting on $\|\psi\rangle$
U_0	000	000	None
U_1	001	000	
		001	S_{w_0,w_1}
U_2	010	000	
		010	S_{w_0,w_2}
U_3	011	000	
		010	S_{w_0,w_2}
		011	S_{w_2,w_3}
U_4	100	000	
		100	S_{w_0,w_4}
U_5	101	000	
		100	S_{w_0,w_4}
		101	S_{w_4,w_5}
U_6	110	000	
		100	S_{w_0,w_4}
		110	S_{w_4,w_6}
U_7	111	000	
		100	S_{w_0,w_4}
		110	S_{w_4,w_6}
		111	S_{w_6,w_7}

3.1.2 The dispositions-programming circuit

Let us define a bijective function $\mathcal{T} : \mathfrak{D}(\mathcal{U}) \to \{0,1\}^{N \log N}$, which assigns one and only one binary string of length $N \log N$ to each possible disposition of the unitary channels belonging to the set $\mathcal{U}$. In particular, $\mathcal{T}$ maps each disposition $U_{i_{(N)}} \ldots U_{i_{(1)}}$ to the string obtained juxtaposing $\mathcal{S}(U_{i_{(N)}}) \ldots \mathcal{S}(U_{i_{(1)}})$.

We are now in position to carry out Task 1 providing the circuit depicted in Eq. (3.5), which is structured as follows:

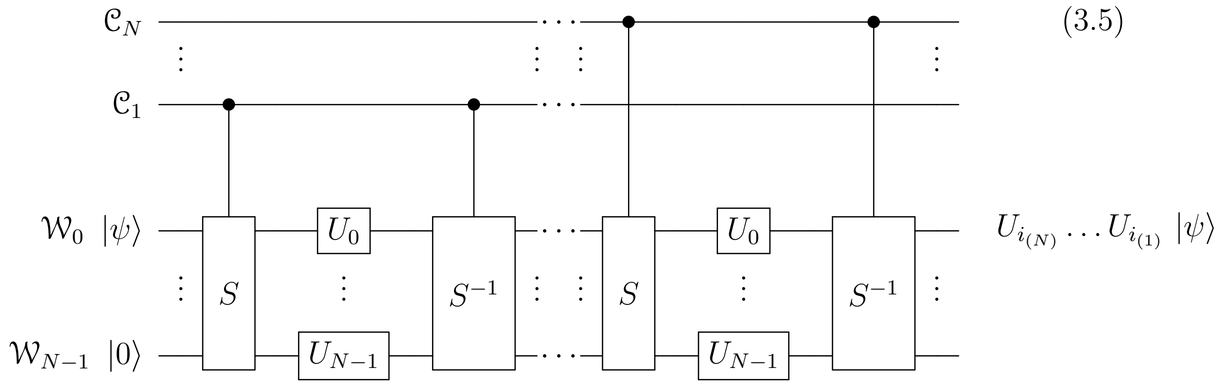
1. A control register, made of $N \log N$ control qubits, into which the programmer inputs a desired state whose representation in the standard basis of $\mathcal{H}^{\otimes N \log N}$ is given by $\mathcal{T}(U_{i_{(N)}} \dots U_{i_{(1)}})$

2. A system register made of N qubits, initially prepared in the state $|\psi\rangle \otimes |0\rangle^{N-1} \in \mathcal{H}^{\otimes N}$, where $|\psi\rangle$ is the relevant state and $|0\rangle$ is a reference state of $\mathcal{H}$.

Let us define the sets of wires $\{\mathcal{C}_i | i = 1, \dots, N\}$ by grouping n neighbouring control qubits at times starting from the first one. The following property characterises the circuit of Eq. (3.5): for every k the set $\mathcal{C}_k$ composed by n control qubits regulates $U_{j_{(k)}}$, namely the kth channel that the system state undergoes.

It is easy to see that one needs $34\, N(N-1)$ elementary operations in order to implement circuit (3.5), as the latter is built up by N Eq.-(3.2)-like juxtaposed circuits.

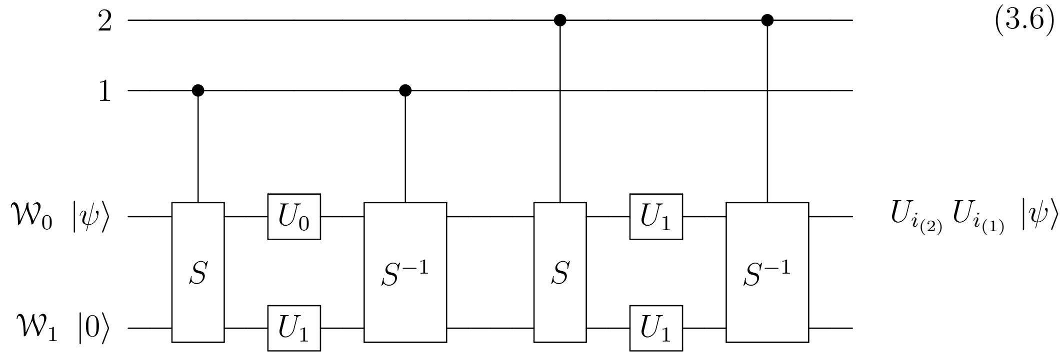
What we said since now allows us to conclude that Task 1 is achievable by an ordinary quantum circuit made of $N \log N$ ancillary qubits plus N system qubits (whose first one is the relevant one and the other are garbage qubits) and resorting to $\mathbf{O}(N^2)$ elementary operations.



(3.5)

Case $N = 2$**,** $n = 1$

The resolution of Task 1 for $N = 2$ is given by the circuit depicted in Eq (3.6). The control register is made by two wires and our goal is accomplished resorting to $34 \cdot 2(2-1) = 68$ elementary operations and utilising each channel twice.



(3.6)

We want to stress that if one feeds the control register with a non classical state (*i.e.* a superposition state) then one gets an entangled state at the end of the computation. We report two particular situations to the reader's attention:

- The control register's state is $|\Phi_1\rangle \propto |01\rangle + |10\rangle$. The state of wire $\mathcal{W}_0$ will consequently be $|\Psi_0\rangle \propto |01\rangle \otimes U_0 U_1 |\psi\rangle + |10\rangle \otimes U_1 U_0 |\psi\rangle$ at the end of the computation. Thus we find all the permutation without repetitions of U_0 and U_1 in the final entangled state.

- The control register's state is $|\Phi_2\rangle \propto |00\rangle + |01\rangle + |10\rangle + |11\rangle$. The state of wire $\mathcal{W}_0$ will consequently be $|\Psi_1\rangle \propto |00\rangle \otimes U_0 U_0 |\psi\rangle + |01\rangle \otimes U_0 U_1 |\psi\rangle + |10\rangle \otimes U_1 U_0 |\psi\rangle + |11\rangle \otimes U_1 U_1 |\psi\rangle$ at the end of the computation. Thus we find all the possible dispositions of U_0 and U_1 in the final entangled state.

As it happens in many other well-known quantum algorithms, what we have just written displays in detail how the power of quantum computation allows one to recover in the final state more than one (and potentially all the) possible outputs of the circuit, each one entangled with the related control qubit.

3.1.3 Optimality

Can we claim that the circuit we provided in Eqs. (3.1) and (3.5) reach their goal employing the minimum number of elementary operations or there is a more efficient strategy with which one can achieve Tasks 2 and 1? The following two theorems ensure that in Sections 3.1.1 and 3.1.2 we gave the optimal solutions to the aforementioned tasks, within the frame of the ordinary quantum circuitry.

Theorem 3.1.1. *The circuit sketched in Eq. (3.2) reaches its goal in the most efficient way within the frame of ordinary quantum circuitry.*

Proof. Let $\mathsf{X} = \{X_1, \ldots, X_N\}$ be a set of N elements. The minimum number of yes-no questions that one must answer to univocally determine an element of the set is given by the *row bit content* of the set X: $r_{\mathsf{X}} = \log |\mathsf{X}|$. Thus the minimum number of yes-no questions that one must answer to single out an element of the set $\mathcal{U}$ introduced in Task 2 is given by $r_{\mathcal{U}} = n$. This exactly corresponds to the number of ancillary wires belonging to the control register of circuit in Eq. (3.2), which we presented as a solution for Task 2 itself. Thus we can conclude that we solved Task 2 using the minimum number of ancillary wires.

Let us now analyse whether the number of Fredkin gates we employed to assemble the circuit in Eq. (3.2) is the minimal one. The best coding algorithm whose goal is to single out an element of a generic set X made of N elements is the following:

- Partition the set X in two halves $\mathsf{X}_0 \doteq \{X_0, \ldots, X_{\frac{N}{2}-1}\}$ and $\mathsf{X}_1 \doteq \{X_{\frac{N}{2}}, \ldots, X_{N-1}\}$;
- Answer the following question: "Which of the two halves X_0 and X_1 does the element X_i belong to?" and retain the subset X_j that contains the sought element;
- For $k = 1, \ldots, n-1$
  - Partition the subset you retained in the previous step in two halves: $\mathsf{X}_{\underbrace{s}_{k}0}$ and $\mathsf{X}_{\underbrace{s}_{k}1}$, likewise it was made in the first step. Here s denotes a string made o of k digits.
  - Answer the following question: "Which of the two halves $\mathsf{X}_{\underbrace{s}_{k}0}$ and $\mathsf{X}_{\underbrace{s}_{k}1}$ does the element X_i belong to?" and retain the subset X_j that contains the sought element.

It is easy to verify that the strategy we adopted in writing Program 2 consists in implementing the best coding algorithm we have reported just before, by encoding the answer of the kth yes-no question in the kth digit of $\mathcal{S}(U_i)$ for every k and for every i and by inserting in the circuit appropriate Fredkin gates that let the state $|\psi\rangle$ reach the desired wire $\mathcal{W}_{\mathcal{S}(U_i)}$.

In particular, the minimal number of Fredkin gates that one must put in the circuit is clearly $N-1$, as the number of wires that the state $|\psi\rangle$ must be able to reach from the starting wire $\mathcal{W}_0$ is indeed $N-1$. Thus we can also claim that circuit of Eq. (3.2) provides us a solution of Task 2 resorting to the minumum number of Fredkin gates.

From what we pointed out in Sec. 3.1.1, it comes that the minimum number of elementary operations one requires to fulfil Task 2 is $\mathbf{O}(N)$. ■

Theorem 3.1.2. *The circuit sketched in Eq. (3.5) reaches its goal in the most efficient way, within the frame of the ordinary quantum circuitry.*

Proof. The proof of this Theorem follows straightforwardly from Theorem 3.1.1.

In fact, since the elements of the string $U_{i_{(N)}} \dots U_{i_{(1)}}$ are independent, the circuit that accomplishes Task 1 in the most efficient way must be build juxtaposing N circuits that optimally achieves Task 2, which is nothing but the circuit sketched in Eq. (3.2), as proved in Theorem 1.

This is just the way we devised in Section 3.1.2 to build up the circuit outlined in Eq. (3.5), which we provided as a solution for Task 1. Thus the the minimum number of elementary operations required to achieve Task 1 is $\mathbf{O}(N^2)$. ■

3.2 Programming permutations of channels through dynamical computational networks

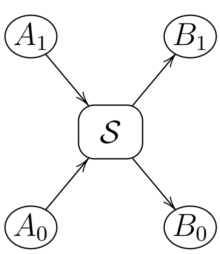
This section is devoted to establish a graphical scheme that suitably describes dynamical computational networks and to solve a task whose requirement is similar to the one of Task 1, in order to see whether dynamical computational network are more powerful than ordinary quantum circuits in programming all the possible permutations of N unitary channels.

3.2.1 Graphical representation of dynamical computational networks

Let us now pick out the main rules that characterise a network in which one or more `QSO`s are displaced. As we have pointed out in Chapter 2, the `QSO` computes a function which is not of type $N \to 1$, thus being not performable through an ordinary quantum circuit.

In order to overcome this drawback, we should devise a new graphical scheme aimed to describe dynamical computational networks.

An effective circuital representation for the `QSO` is provided by the following graph:



(3.7)

Three kinds of vertexes appear in the graph of Eq. (3.7):

- A_i-kind vertexes with no input arrows and one output arrow, representing the teeth of the no-signalling channel inputting the QSO;
- B_j-kind vertexes with one input arrow and no output arrows, representing the teeth of the no-signalling channel outputting the QSO;
- The $\mathcal{S}$-kind vertex with two input arrows and two output arrows, that represents the device performing the function whose Choi-Jamiołkowski operator was given in Eq. (2.38).

For sake of simplicity in the representation, we missed out in Eq. (3.7) the wire associated to the control qubit that rules the functioning of the QSO. We stress that the arrows in Eq. (3.7) do not denote qubits, but they stand for channels.

3.2.2 Assemblage of the most efficient permutations-programming network

We now investigate the possibility of programming all the possible permutations of N unitary channels and superpositions of them through a dynamical computational network, exploiting the graph representation we have settled on in the previous subsection. More precisely, we try to achieve the following:

Task 3. *Let $\mathfrak{U} \doteq \{U_i \,|\, i = 0, \ldots, N-1\}$ be a set of N unitary channels defined on the space $\mathcal{H}$ of a qubit system. Build an efficient dynamical computational network which is able to program one of the $N!$ permutations without repetitions of the N above introduced channels, depending on the state of a control register.*

We now provide a constructive solution to Task 3, which will also be proven to be the efficient one. The graph whose sequence of output vertexes $\{B_j \,|\, j = 0, \ldots, N-1\}$ is ordered as one of the possible permutation without repetition of the channels U_i is assembled according to the following program:

`Program 3 (PERMUTATIONS-PROGRAMMING NETWORK)`

- Take the graph in Eq. (3.7) with $\begin{cases} A_0 = U_0 \\ A_1 = U_1 \end{cases}$ and name $\mathcal{S}_1^{(1)}$ the only $\mathcal{S}$-kind vertex. Let it be controlled by the wire $s_1^{(1)}$.
- `FOR (k = 2, k = n-1, k++)`
    - Introduce the k QSOs $\mathcal{S}_1^{(k)}, \ldots, \mathcal{S}_k^{(k)}$;

- Substitute the vertexes $\begin{cases} B_0 & \text{with} & \mathcal{S}_1^{(k)} \\ & \vdots & \\ B_k & \text{with} & \mathcal{S}_k^{(k)} \end{cases}$;

- Establish the connections $\mathcal{S}_{i+1}^{(k)} \to \mathcal{S}_i^{(k)} \quad (i = 1, \ldots, k-1)$ and let each vertex $\mathcal{S}_i^{(k)}$ be controlled by the wire $s_i^{(k)}$;

- Introduce the k B_j-kind vertexes $B_0, \ldots, B_k$;

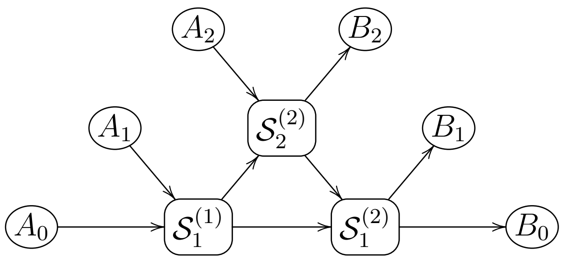
- Establish the further connections $\begin{cases} \mathcal{S}_1^{(k)} & \to & B_0 \\ \mathcal{S}_1^{(k)} & \to & B_1 \\ \mathcal{S}_2^{(k)} & \to & B_2 \\ & \vdots & \\ \mathcal{S}_k^{(k)} & \to & B_k \end{cases}$.

Let us now tally the number of `QSO`s that one needs to reach its goal when $|\mathcal{U}| = N$. Only one `QSO` is needed in the first step of Program 3 whereas k of them are required in the kth step of the `FOR` cycle. Thus Program 3 needs overall

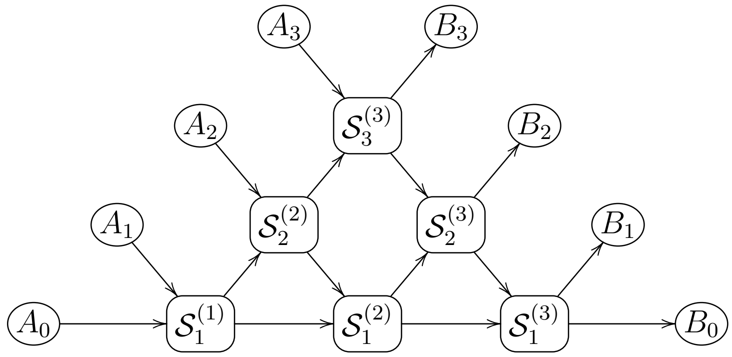
$$\sum_{n=1}^{N-1} n = \frac{1}{2} N(N-1)$$

`QSO`s to reach its goal if $|\mathcal{U}| = N$.

The graphs emerging in the cases $N = 3$ and $N = 4$ are reported in Eqs. (3.8) and (3.9), in which the control wires are omitted on the same aforementioned grounds.



(3.8)



(3.9)

Table 3.3 lists all the six possible outputs of the circuit and provides for each of them the corresponding control state(s). Notice that more than one control register's state can be linked to the same output.

Table 3.3

$B_0\, B_1\, B_2$	State of		
	$s_1^{(1)}$	$s_1^{(2)}$	$s_2^{(2)}$
$U_0\, U_1\, U_2$	$\|0\rangle$ $\|1\rangle$	$\|0\rangle$ $\|1\rangle$	$\|0\rangle$ $\|0\rangle$
$U_1\, U_2\, U_0$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$U_2\, U_0\, U_1$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$U_0\, U_2\, U_1$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
$U_1\, U_0\, U_2$	$\|1\rangle$ $\|0\rangle$	$\|0\rangle$ $\|1\rangle$	$\|0\rangle$ $\|0\rangle$
$U_2\, U_1\, U_0$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$

Program 3 reaches its goal in the more efficient way, as it will be proved in the following

Theorem 3.2.1. *Program 3 realises its goal in the most efficient way, namely resorting to the minimum number of* `QSO`*s and to the minimum number of ancillary control wires.*

Proof. The `QSO` is a computational resource which takes as input two channels and answers the following question: "Which is the mutual ordering of the two channels in the output string $B_0 \dots B_{N-1}$?" The answer to this question is obviously a yes-no-kind answer: either U_i precedes U_j or *vice versa* U_j precedes U_i in the output string.

The target of Program 3 consists in establishing the mutual ordering of the input channels depending on the state of the control register, through assembling a dynamic computational network made of `QSO`s. Thus the number of questions that are supposed to be answered in that program equals the number of all the possible couples that can be formed from the N input channels $(A_i,\, A_j)_{i\neq j}$. Since the number of different couples that can be formed from N objects is $\frac{1}{2}N(N-1)$ and this is also the number of `QSO` and of ancillary wires that Program 3 requires in order to achieve its aim when $|\mathcal{U}| = N$, then the same program realises its goal in the most efficient way. ■

Unlike the situation we analysed in Sec. 3.1, the potentialities of a dynamical computational network allow one to program any permutation of N channels with only one use of each channel. Nevertheless a question raises comparing Theorems 3.1.2 and 3.2.1: Why the ancillary wires required to achieve Task 1 are more than the ones required to achieve Task 2? The answer to this question lies in the different ways in which the circuit of Eq. (3.6) and the dynamic network emerging from Program 3 are built up for fixed N. In fact – as it was stressed in Sec. 3.1.2 – the circuit of Eq. (3.6) is assembled resorting to N Eq.-(3.2)-like juxtaposed *independent* circuits. Instead a dynamic computational network with N inputs emerging from Program 3 requires at each step of the computation items of information about the mutual ordering of the channels that have been already inputted in the previous steps of the computation.

Conclusions

We now sum up the original results that we obtained in this thesis. The first remarkable result is the establishment of a pictorial representation for dynamic computational networks, namely for those networks in which any `QSO` is included. Tackling the problem of programming dispositions and permutations of N unitary channels and superpositions of them, we achieved the following results:

- Within the framework of ordinary quantum circuitry one can implement a multiple controlled swap gate $\mathrm{C}^{(n)}(S)$ acting on a N qubit register resorting to $\mathbf{O}(N)$ elementary operations.

- Within the framework of ordinary quantum circuitry one can program all the dispositions of N unitary channels and superpositions of them utilising each channel N times and resorting to $\mathbf{O}(N^2)$ elementary operations, having at disposal an ancillary register made up of $N \log N$ wires.

- Within the framework of dynamic computational networks one can program all the permutations of N unitary channels and superpositions of them utilising each channel once and resorting to $\mathbf{O}(N^2)$ elementary operations, having at disposal an ancillary register made up of $\frac{1}{2}N\,(N-1)$ wires.

What we discovered does not allow us to state whether the computational power of dynamical computational networks is stronger than the one proper of ordinary quantum circuits. The issue of looking for a task that is hard to solve through an ordinary quantum computer and yet tractable implementing an appropriate dynamical computational network still remains an open problem. Moreover we cannot establish the truth of Conjecture 1 from our results.

Taking inspiration from Ref. [29], we wish to end the present work by posing a question: "Can we say that any map which is in principle admissible according to quantum mechanics is also feasible?" Put in another way, is there always any experimental set-up with which one could realise any admissible map? If Conjecture 1 is true, the answer to this question will be affirmative if one is able to experimentally implement the `QSO`. Some efforts to reach this last goal have been carrying on by Hall, Altepeter and Kumar in [44], but nowadays this is another open problem.

Appendix A

Mathematical preliminaries

This appendix is intended to fix the notation of the present work and to give the reader coming from another major field of studies sufficient preliminaries about linear maps and Choi-Jamiołkowski operators.

A.1 The Choi-Jamiołkowski isomorphism

Finite dimensional complex Hilbert spaces are denoted by $\mathcal{H}$, with a label when we need to distinguish them, as

$$\mathcal{H}_0, \mathcal{H}_1, \dots \tag{A.1}$$

A vector ψ belonging to a Hilbert space $\mathcal{H}_i$ will be indicated with the "ket" notation $|\psi\rangle_i$. We will denote with $\mathcal{L}(\mathcal{H})$ the space of linear operator on a Hilbert space $\mathcal{H}$. The space of linear operators from $\mathcal{H}_0$ to $\mathcal{H}_1$ will be denoted by $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$. Sometimes, especially in applications, this notation will be slightly modified, in particular it is convenient to indicate Hilbert spaces with a roman letter (such as A,B, ...), in order to avoid the notational overburden of many numerical indices.

In the following we will always assume that any d-dimensional Hilbert space $\mathcal{H}$ is given with some fixed orthonormal basis $|n\rangle, n = 0, \dots, d-1$, such that we can identify

$$\mathcal{H} \cong \mathbb{C}^d, \tag{A.2}$$

Moreover, we can identify an operator $A \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ with a complex matrix

$$A_{nm} \doteq {}_1\langle n| A |m\rangle_0 . \tag{A.3}$$

To express the well-known isomorphism

$$\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1) \cong \mathcal{H}_1 \otimes \mathcal{H}_0 \tag{A.4}$$

we will use the following explicit "double ket" correspondence

$$\begin{aligned} A \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1) \longleftrightarrow |A\rangle\!\rangle_{0,1} &\doteq (A \otimes I_{\mathcal{H}_0})|I_{\mathcal{H}_0}\rangle\!\rangle \\ &= (I_{\mathcal{H}_1} \otimes A^T)|I_{\mathcal{H}_1}\rangle\!\rangle \end{aligned} \tag{A.5}$$

where $|I_{\mathcal{H}}\rangle\!\rangle \doteq \sum_{n=0}^{\dim(\mathcal{H})-1} |n\rangle\,|n\rangle$, and the transposition is made with respect to the fixed orthonormal bases.

Combining this isomorphism with the isomorphism $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0) \cong \mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_0)$ we also obtain a third fundamental isomorphism between the space of linear maps from $\mathcal{L}(\mathcal{H}_0)$ to $\mathcal{L}(\mathcal{H}_1)$, and linear operators $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$:

$$\mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1)) \cong \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0). \tag{A.6}$$

The explicit correspondence is given by the following

Definition A.1.1 (Choi-Jamiołkowski isomorphism)**.** *The* Choi-Jamiołkowski isomorphism *is a bijection*

$$\mathfrak{C} : \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1)) \longrightarrow \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0) \tag{A.7}$$

which, for every map $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1))$ gives the following operator $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$

$$M = \mathfrak{C}(\mathcal{M}) \doteq \mathcal{M} \otimes \mathcal{I}_{\mathcal{L}(\mathcal{H}_0)}(|I_{\mathcal{H}_0}\rangle\!\rangle\langle\!\langle I_{\mathcal{H}_0}|). \tag{A.8}$$

The inverse transformation $\mathfrak{C}^{-1}$ defines a map $\mathfrak{C}^{-1}(M)$ acting on $\mathcal{L}(\mathcal{H}_0)$ as follows

$$\mathcal{M}(X) = \mathfrak{C}^{-1}(M)(X) = \operatorname{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^T)M]. \tag{A.9}$$

Lemma A.1.1. *A linear map $\mathcal{M}$ is trace-preserving if and only if its Choi-Jamiołkowski operator enjoys the property*

$$\operatorname{Tr}_{\mathcal{H}_1}[M] = I_{\mathcal{H}_0}. \tag{A.10}$$

Proof. The trace preserving condition is $\operatorname{Tr}[\mathcal{M}(X)] = \operatorname{Tr}[X]$. Since

$$\operatorname{Tr}[\mathcal{M}(X)] = \operatorname{Tr}[(I_{\mathcal{H}_1} \otimes X^T)M] = \operatorname{Tr}_{\mathcal{H}_0}[X^T \operatorname{Tr}_{\mathcal{H}_1}[M]], \tag{A.11}$$

and $\operatorname{Tr}[X] = \operatorname{Tr}[X^T]$, the trace-preserving condition is satisfied for arbitrary X if and only if $\operatorname{Tr}_{\mathcal{H}_1}[M] = I_{\mathcal{H}_0}$. ■

Lemma A.1.2. *A linear map $\mathcal{M}$ is Hermitian preserving if and only if its Choi-Jamiołkowski operator M is Hermitian.*

Proof. A map $\mathcal{M}$ is Hermitian preserving if $\mathcal{M}(H)^\dagger = \mathcal{M}(H)$ for any Hermitian operator H. Equivalently, if $\mathcal{M}(X^\dagger) = \mathcal{M}(X)^\dagger$ for any operator X. We have that

$$\mathcal{M}(X)^\dagger = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^*)M^\dagger] = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^{\dagger T})M^\dagger]. \tag{A.12}$$

Clearly, if $M^\dagger = M$ one has $\mathcal{M}(X)^\dagger = \mathcal{M}(X^\dagger)$. On the other hand, if

$$\mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^{\dagger T})M^\dagger] = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^{\dagger T})M] \tag{A.13}$$

for all X, then $M^\dagger = M$, due to the Choi-Jamiołkowski isomorphism. ∎

Lemma A.1.3. *A linear map $\mathcal{M}$ is completely positive (CP) if and only if its Choi-Jamiołkowski operator M is positive semidefinite.*

Proof. Clearly, if $\mathcal{M}$ is CP, by Eq. (A.8) $M \geqslant 0$. On the other hand, if $M \geqslant 0$, it can be diagonalized as follows

$$M = \sum_j |K_j\rangle\rangle\langle\langle K_j|, \tag{A.14}$$

and consequently, exploiting Eqs. (A.9) and (A.5), we can write its action in the Kraus form [19]

$$\mathcal{M}(X) = \sum_j K_j X K_j^\dagger. \tag{A.15}$$

The Kraus form coming from diagonalization of M is called *canonical*. On the other hand, since the same reasoning holds for any decomposition $M = \sum_k |F_k\rangle\rangle\langle\langle F_k|$, there exist infinitely many possible Kraus forms. The Kraus form implies complete positivity: indeed, the extended map $\mathcal{M} \otimes \mathcal{I}_{\mathcal{L}\mathcal{H}_A}$ transforms any positive operator $P \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_A)$ into a positive operator, as follows

$$\mathcal{M} \otimes \mathcal{I}_{\mathcal{L}(\mathcal{H}_A)}(P) = \sum_j (K_j \otimes I_{\mathcal{H}_A}) P (K_j^\dagger \otimes I_{\mathcal{H}_A}) \geqslant 0. \tag{A.16}$$

∎

A.2 The link product

The Choi-Jamiołkowski isomorphism poses the natural question on how the composition of linear maps is translated to a corresponding composition between the respective Choi-Jamiołkowski operators.

Consider two linear maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1))$ and $\mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{H}_2))$ with Choi-Jamiołkowski operators $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $N \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$, respectively. The two maps are composed to give the linear map $\mathcal{C} = \mathcal{N} \circ \mathcal{M} \in$

$\mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_2))$. This can be easily obtained upon considering the action of $\mathcal{C}$ on an operator $X \in \mathcal{L}(\mathcal{H}_0)$ written in terms of the Choi-Jamiołkowski operators of the composing maps

$$\begin{aligned} \mathcal{C}(X) &= \mathrm{Tr}_{\mathcal{H}_1}[(I_{\mathcal{H}_2} \otimes \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^T)M]^T)N] \\ &= \mathrm{Tr}_{\mathcal{H}_1,\mathcal{H}_0}[(I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_1} \otimes X^T)(I_{\mathcal{H}_2} \otimes M^{T_1})(N \otimes I_{\mathcal{H}_0})]. \end{aligned} \tag{A.17}$$

Upon comparing the above identity with the Eq. (A.9) for the map $\mathcal{C}$, namely $\mathcal{C}(X) = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_2} \otimes X^T)C]$, one obtains

$$C = \mathrm{Tr}_{\mathcal{H}_1}[(I_{\mathcal{H}_2} \otimes M^{T_1})(N \otimes I_{\mathcal{H}_0})], \tag{A.18}$$

where M^{T_i} denotes the partial transpose of M on the space $\mathcal{H}_i$. The above result can be expressed in a compendious way by introducing the notation

$$N * M \doteq \mathrm{Tr}_{\mathcal{H}_1}[(I_{\mathcal{H}_2} \otimes M^{T_1})(N \otimes I_{\mathcal{H}_0})], \tag{A.19}$$

which we call *link product* of the operators $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $N \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$. The above result can be synthesized in the following statement.

Theorem A.2.1 (Composition rules)**.** *Consider two linear maps*

$$\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1)) \tag{A.20}$$

and

$$\mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{H}_2)) \tag{A.21}$$

with Choi-Jamiołkowski operators $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $N \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$, respectively. Then, the Choi-Jamiołkowski operator $C \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_0)$ of the composition $\mathcal{C} = \mathcal{N} \circ \mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_2))$ is given by the link product of the Choi-Jamiołkowski operators $C = N * M$.

In the following we will consider more generally maps with input and output spaces that are tensor products of Hilbert spaces, and which will be composed only through some of these spaces, *e.g.* for quantum circuits which are composed only through some wires. For describing these compositions of maps we will need a more general definition of link product. For such purpose, consider now a couple of operators $M \in \mathcal{L}(\bigotimes_{m\in \mathsf{M}} \mathcal{H}_m)$ and $N \in \mathcal{L}(\bigotimes_{n\in \mathsf{N}} \mathcal{H}_n)$, where M and N describe sets of indices for the Hilbert spaces, which generally have nonempty intersection.

The general definition of link product then reads:

Definition A.2.1 (General link product)**.** *The* link product *of two operators $M \in \mathcal{L}(\bigotimes_{m\in \mathsf{M}} \mathcal{H}_m)$ and $N \in \mathcal{L}(\bigotimes_{n\in \mathsf{N}} \mathcal{H}_n)$ is the operator $M * N \in \mathcal{L}(\mathcal{H}_{\mathsf{N}\setminus\mathsf{M}} \otimes \mathcal{H}_{\mathsf{M}\setminus\mathsf{N}})$ given by*

$$N * M \doteq \mathrm{Tr}_{\mathsf{M}\cap\mathsf{N}}[(I_{\mathsf{N}\setminus\mathsf{M}} \otimes M^{T_{\mathsf{M}\cap\mathsf{N}}})(N \otimes I_{\mathsf{M}\setminus\mathsf{N}})], \tag{A.22}$$

where the set-subscript X is a shorthand for $\bigotimes_{i\in\mathsf{X}} \mathcal{H}_i$, and $\mathsf{A}\setminus\mathsf{B} \doteq \{i | i \in \mathsf{A}, i \notin \mathsf{B}\}$ for two sets A and B.

Examples. For $\mathsf{M} \cap \mathsf{N} = \emptyset$, *e.g.* for two operators M and N acting on different Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_0$, respectively, their link product is the tensor product:

$$N * M = N \otimes M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0). \tag{A.23}$$

For $\mathsf{N} = \mathsf{M}$, *i.e.* when the two operators M and N act on the same Hilbert space, the link product becomes the trace

$$A * B = \mathrm{Tr}[A^T B]. \tag{A.24}$$

Theorem A.2.2 (Properties of the link product)**.** *The operation of link product has the following properties:*

1. *$M * N = E(N * M)E$, where E is the unitary swap on $\mathcal{H}_{\mathsf{N}\setminus\mathsf{M}} \otimes \mathcal{H}_{\mathsf{M}\setminus\mathsf{N}}$.*
2. *If M_1, M_2, M_3 act on Hilbert spaces labelled by the sets $\mathsf{I}_1, \mathsf{I}_2, \mathsf{I}_3$, respectively, and $\mathsf{I}_1 \cap \mathsf{I}_2 \cap \mathsf{I}_3 = \emptyset$, then $M_1 * (M_2 * M_3) = (M_1 * M_2) * M_3$.*
3. *If M and N are Hermitian, then $M * N$ is Hermitian.*
4. *If M and N are positive semidefinite, then $M * N$ is positive semidefinite.*

Proof. Properties 1, 2, and 3 are immediate from the definition. For Property 4, consider the two maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathsf{M}\setminus\mathsf{N}}), \mathcal{L}(\mathcal{H}_{\mathsf{M}\cap\mathsf{N}}))$ and $\mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathsf{M}\cap\mathsf{N}}), \mathcal{L}(\mathcal{H}_{\mathsf{N}\setminus\mathsf{M}}))$, associated to M and N by Eq. (A.9). Due to Lemma A.1.3, the maps $\mathcal{M}$ and $\mathcal{N}$ are both CP. Moreover, due to Theorem A.2.1 the link product $C = N * M$ is the Choi-Jamiołkowski operator of the composition $\mathcal{C} = \mathcal{N} \circ \mathcal{M}$. Since the composition of two CP maps is CP, the Choi-Jamiołkowski operator $C = N * M$ must be positive semidefinite. ■

As it should be clear to the reader, the advantage in using multipartite operators instead of maps is that we can associate many different kinds of maps to the same operator $M \in \mathcal{L}(\bigotimes_{i\in I} \mathcal{H}_i)$, depending on how we group the Hilbert spaces in the tensor product. Indeed, any partition of the set I into two disjoint sets I_0 and I_1 defines a different linear map from $\mathcal{L}(\bigotimes_{i\in I_0} \mathcal{H}_i)$ to $\mathcal{L}(\bigotimes_{i\in I_1} \mathcal{H}_i)$ via Eq. (A.9).

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